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Section 1

Literature Review on Mechanical Reliability and Probabilistic Design

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1. INTRODUCTION

A simple illustration of the basic problem of structural or mechanical reliability (or "probabilistic design") is provided in Fig. 1. The problem is to ensure that the probability of failure of the cantilever beam is acceptably small under the action of the stochastic load, $Q(t)$. Assume that localized yielding defines failure. The probability of failure p_f is then the probability of the event that the maximum stress, S, corresponding to the maximum load, Q, (assuming that dynamics is not important) exceeds the yield strength, R.

$$
\begin{aligned}
p_f &= P(R \le S) \\
&= P\left(R \le \frac{6QL}{bh^2}\right)
\end{aligned}
\tag{1}
$$

Assume all factors (R, Q, L, b, h) possess uncertainty and are modelled as random variables. It follows from the definition of the joint probability density function that

$$
p_f = P(R \le S) = \int_{\Omega} f_{\underset{\sim}{X}}(\underset{\sim}{x})\,d\underset{\sim}{x} \tag{2}
$$

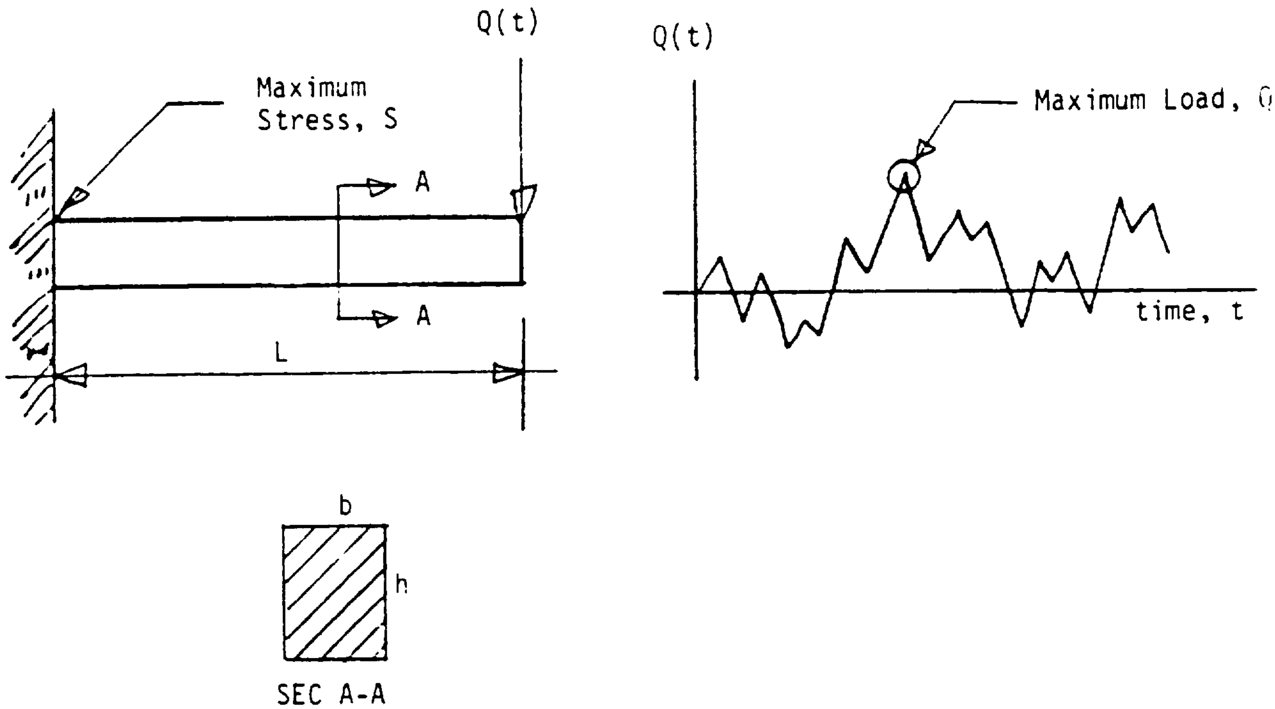
where, in general, $\underset{\sim}{X}$ is the vector of all design factors, and $f_{\underset{\sim}{X}}$ is the joint probability density function of the random design factors. Ω is the region of failure, i.e., where $R \le S$.

Relative to the PSAM project, a similar formulation can be used to construct a cumulative distribution function (cdf) of a stress or a response, U. Assume that U is a function of several design factors.

$$
U = f(\underset{\sim}{X}) \tag{3}
$$

The cdf of U is defined as

$$
F_U(u) = P(U \le u) \tag{4}
$$



- Maximum Stress $S = \frac{MC}{I} = \frac{6QL}{bh^2}$
- Strength of material ≡ yield strength, R
- Event of failure (R < S)

Fig. 1 An illustration of a simple design problem in which all design factors (R, Q, L, b, and h) can be considered as random variables.

By analogy, $F_F(u)$ can be evaluated by Eq. 2 where $\Omega = (U \leq u)$.

In the general case, solution of the multi-dimensional integral of Eq. 2 is impossible in practice. Development of "probabilistic design theory" (or "structural reliability") is directed towards practical solution to problems of the type of Eq. 1 for the purpose of (a) reliability assessment of existing designs, and (b) development of probability based design requirements.

Presented here is a narrative summary of literature in mechanical reliability (probabilistic design). Contributions are growing and this presentation is not comprehensive. However, there is confidence that most of the important works are cited. The emphasis in this review is for application to the PSAM project.

2. HISTORICAL NOTE

The origin of modern probability theory dates back to the 17th century when an ardent gambler, Chevalier de Mére consulted the Franch mathematician Blaise Pascal (1623-1662) regarding a problem about a game of chance. Pascal, in turn, corresponded with Pierre Fermat (1601-1665). Subsequently, there was a rapid growth in interest in the mathematics of probability applied to games of chance. Karl Gauss (1777-1855) and Pierre Laplace (1749-1827) were the first to find applications in other fields. But serious interest in the systematic application of probabilistic and statistical methods to structural and mechanical design did not develop until the mid-1950's.

A brief history of the development of the theory of structural reliability is presented in the text by Lind, Krenk, and Madsen [L5] and in the Ph.D dissertation of Kjerengtroen [K3]. Parts of the following are quoted from their work. The history of structural reliability goes back some 50-60 years. The first phase appears in retrospect as a very slow beginning. Early pioneering contributions included those of Forsell [F5] and Mayer [M2], and later Basler [B1].

M. Prot published several papers (in French) from 1936 to 1951 on Statistical distribution of stresses. And Weibull developed statistical theories of strength; his name is now associated with an extreme value distribution of minima [W1, W2]. Later Pugsley [P6] and Johnson [J1] gave comprehensive presentations on the theory of structural reliability and of economical design.

The modern era of probabilistic mechanical design started after the Second World War. In October 1945, a paper entitled, "The Safety of Structures" appeared in the proceedings of the American Society of Civil Engineers. This historical paper was written by A. M. Freudenthal, and the purpose of it was to "analyse the safety factor in engineering structures in order to establish a rational method of evaluating its magnitude." It was selected for inclusion with many discussions in the 1947 Transactions of the American Society of Civil Engineers [F6]. The publication of this paper marked the genesis of structural reliability in the U.S. Most of the ingredients of structural reliability such as probability theory, statistics, structural analysis and design, quality control, existed prior to that time. Nevertheless, Prof. Freudenthal was the first to put them together in a definitive and comprehensive manner. He continued, for many years, to be in the forefront of structural reliability and risk analysis as well as fatigue and fracture studies. A sample of his significant publications on structural reliability and fatigue are provided in Refs. F7 and F8. Another landmark paper in structural reliability which began to formalize analysis was written by Freudenthal, Garrelts, and Shinozuka and was published in 1966 [F9].

During the 1960's there was rapid growth of academic interest in structural reliability theory. Classical theory became well developed and widely known through a few influential publications such as that of Freudenthal, Garrelts, and Shinozuka [F9], Pugsley [P5], Kececioglu and

Cormier [K2], Ferry-Borges and Castenheta [F2], and Haugen [H3]. However, professional acceptance was low for several reasons. Probabilistic design seemed cumbersome, the theory seemed intractible mathematically and numerically. Little data were available. Modelling error was unknown. And system structural safety analyses seemed extraordinarily complex.

The early 1960's were spent in the search to circumvent these difficulties. Turkstra [T2] presented structural design as a problem of decision making under uncertainty and risk. Lind, Turkstra, and Wright [L2] define the problem of rational design of a code as finding a set of best values of the load and resistance factors. Cornell [C2] suggested the use of a second moment format, and subsequently it was demonstrated that Cornell's safety index requirement could be used to derive a set of safety factors on loads and resistance. This approach related reliability analysis to practically accepted methods of design. It has been modified and employed in many structural standards.

In the ensuing years some serious difficulties with the second moment format were discovered in the development of practical examples. First, it was not obvious how to define a reliability index in cases of multiple random variables, e.g., when more than two loads were involved. More disturbingly, Ditlevsen [D2] and Lind [L3] independently discovered the problem of invariance. Cornell's index was not constant when certain simple problems were reformulated in a mechanical equivalent way. Several years were spent in the search of a way out of the dilemma without resolution. In the early 1970's, therefore, second moment reliability based structural design was becoming widely accepted although at the same time it seemed impossible to develop a logically firm basis for the rationale.

The logical impasse of the invariance problem was overcome in the early 1970's. Hasofer and Lind [H2] defined a generalized safety index which was invariant to mechanical formulation. This landmark paper represented a turning point in structural reliability theory. Contributions, proposed in recent years, are extentions of the Hasofer-Lind approach which are more sophisticated mathematically. The era of modern probabilistic design theory which extends from the early 1970's to the present is reviewed in Section 6.

3. GENERAL REFERENCE TEXTS ON PROBABILITY THEORY AND MATHEMATICAL STATISTICS

There are a large number of text books on the market with the approximate title of, "Probability and Statistics for Engineers and Scientists." Two which are recommended for easy reading and reference are those by Meyer [M3] and Hines and Montgomery [H5]. At a slightly higher level is the work of Bowker and Lieberman [B4]. Texts on the same level (upper class undergraduate) written by engineers for engineering practice include those of Benjamin and Cornell [B3] and the two volumes by Ang and Tang [A2, A3]. Intermediate texts on mathematical statistics include those by Freund [F10], Mood and Graybill [M4], and Lindgren [L6]. These present advanced topics, but in a form which can be understood by engineers having some background.

Designs are often selected on the basis of extreme loads stresses or strains. Extreme value theory is described in most of the above references, e.g., both Mood and Graybill [M4] and Lindgren [L6] have discussions on order statistics. And and Tang's second volume has a chapter on extreme value theory [A3]. The elementary text by Hahn and Shapiro [H1] on statistical models in engineering, has a good elementary description of extreme value theory. However, the most definitive work on this topic, although it is difficult reading, is the text by Gumbel published in 1958 [G4].

4. MECHANICAL RELIABILITY AND PROBABILISTIC DESIGN TEXT BOOKS AND GENERAL REFERENCES

There are a few text books which provide elementary information on basic probabilistic theory and application to mechanical design. These include texts by Kapur and Lamberson [K1] and Siddall [S2]. Haugen has written two books on probabilistic mechanical design, the first was published in 1968 [H3] and the second appeared in 1980 [H4]. These texts provide a wealth of practical information, but all fail to provide comprehensive summaries of modern techniques of reliability analyses developed in the past ten years.

Two other texts which are very useful for many engineering applications are those of Mann et al. [M1] and Lipson and Sheth [L4]. The former focuses upon classical reliability models and is an excellent reference for general applications, . . . but no design theory. Lipson and Sheth have much useful information not considered elsewhere.

Advanced text books which treat modern reliability theory include that of Elishakoff [E1], Leporati [L1], and Ang and Tang [A3]. A new text, not yet published, by Lind, Krenk, and Madsen [L5] is an advanced work which summarizes the mathematical theory of structural reliability. At this time, however, perhaps the most highly regarded text is that of Thoft-Christensen and Baker [T1].

Other references which provide general summaries of modern design theory include works by Shinozuka [S1], Wirsching [W5] in addition to CIRIA 63 report [R3], and the NBS report of Ellingwood et al. [E2].

5. CONFERENCE PROCEEDINGS AND PERIODICALS

In recent years there have been a number of specialty conferences on structural reliability. The International Conference on Structural Safety and Reliability ICCOSAR is held every three years, but conference proceedings are not readily available. The 2nd International Conference on Code Formats in 1976 was a particularly productive one and its proceedings are somewhat of a classic [D1]. The ASCE has sponsored a series of four specialty conferences since 1969 entitled, Probabilistic Mechanics and Structural Reliability. Proceedings are available through ASCE for the 1979 and 1984 conferences [P1, P3]. ASCE has also sponsored a specialty conference in 1981 and has published conference proceedings entitled, Probabilistic Methods in Structural Engineering which contain some excellent summary articles [P2].

There is a new journal entitled, Structural Safety (published by Elsevier) strictly dedicated to structural reliability. In addition, the civil engineering profession has been perhaps most active in the development of modern structural reliability concepts and the ASCE Journal of Engineering Mechanics and Journal of Structural Engineering contain almost monthly articles on probabilistic design theory. Survey and theme articles published in the Journal of Structural Engineering include a literature review on structural safety published in 1972 [S4], a series of six articles in 1974 [S5], a series of eight articles in load and resistance factor design in 1978 [G2], and a series of four articles in fatigue reliability in 1982 [F1]. Moreover, ASCE committees (e.g., the ASCE Administrative Committee on Structural Safety and Reliability, . . . and its five working committees) have sponsored many technical sessions and produced numerous articles.

6. A NARRATIVE SUMMARY OF THE DEVELOPMENT AND IMPORTANT REFERENCES OF "MODERN" MECHANICAL RELIABILITY ANALYSIS.

6.1 Mean Value First Order Second Moment Methods (MVFOSM)

The beginning of modern probabilistic design theory can be arbitrarily defined by the introduction of the safety index by Cornell in 1969 [C3]. First define the "failure function," Z, or "limit state," so that the event $Z \leq 0$ is failure. For the example of Eq. 1

$$
\begin{aligned}
Z &= R - S \\
&= R - \frac{6QL}{bh^2}
\end{aligned}
\tag{5}
$$

In general, Z will be a function of k random design factors, X_i. An approximation to the mean value of Z, μ_Z, and standard deviation of Z, σ_Z, can be derived using the first terms of a Taylor's series expansion.

$$
\mu_Z = Z(\underset{\sim}{\mu}) \tag{6}
$$

$$
\sigma_Z^2 = \sum_{i=1}^{k} \left(\frac{\partial Z}{\partial X_i} \right)_{\underset{\sim}{\mu}}^2 \sigma_i^2 \tag{7}
$$

where μ_i and σ_i are the mean and standard deviations of X_i respectively. $\underset{\sim}{\mu}$ is the vector of mean values, (as a rule of thumb, higher order terms are significant if the COV's of the variables are greater than 15%).

The safety index defined by Cornell is

$$
\beta = \frac{\mu_Z}{\sigma_Z} \tag{8}
$$

In the special case where Z is linear in normal variates (or Z is a multiplicative function in only lognormal variates), the probability of failure is exactly

$$
p_f = \Phi(-\beta) \tag{9}
$$

For all other cases the probability of failure so computed is only approximate. In some cases, the approximation is not bad; in other cases the error is very significant.

In complicated problems it is often relatively easy to numerically estimate μ_Z and σ_Z. This may be particularly the case in the PSAM project in which the random variables are related to a computer algorithm. Using a simple straightforward perturbation technique, the derivatives of Eq. 7 can be computed.

6.2 Normal and Lognormal Formats

Before consideration of advanced methods, the two special cases where β defines the exact reliability should be noted. Assume that the failure function is of the form

$$Z = A_0 + \sum_{i=1}^{m} A_i X_i \tag{10}$$

where all X_i are normal and the A's are constant. Z is normal, and the μ_Z and σ_Z of Eqs. 6 and 7 are exact as is p_f of Eq. 9. This is known as the "normal format." In many cases, Haugen has shown that mechanical design problems can be approximated by the normal format [H3, H4].

Another common form seen in design is the multiplicative function in which the failure event can be written as

$$A \prod_{i=1}^{n} X_i^{a_i} \geq 1 \tag{11}$$

Here A and the a_i's are constant. By taking the log of both sides of Eq. 11, the linear form of Eq. 10 results, . . . along with the condition that failure is defined by $Z \leq 0$. And if all X_i are lognormal, then Eq. 9

given an exact form for p_f. This "lognormal format" which forms the analytical foundation for LRFD [G2], is employed by Wirsching for fatigue reliability calculations [W5, W7, W8].

Because a closed form expression for probability results from the normal or lognormal format, assumptions should be made wherever practical in the PSAM project to produce this simplified form.

6.3 Advanced Reliability Methods; The Generalized Safety Index

The failure function, Z, of Cornell is defined so that failure is the event that $Z \leq 0$. So defined, the algebraic formulation of Z is not unique. For example, it would be equally valid to write the Z of Eq. 5 as

$$Z = \frac{Rbh^2}{6L} - Q \tag{12}$$

A fatal flaw in the safety index of Eq. 8 is that β is not invariant to mechanical formulation of the failure function, e.g., the β of Eq. 8 would depend upon whether Eq. 5 or Eq. 12 was used.

In 1973 Hasofer and Lind presented a new definition of the safety index data which overcame the lack of invariance problem [H2]. The scheme works like this. Each "basic design variable" X_i is transformed by subtracting its mean, μ_i, and dividing by its standard deviation σ_i. The "reduced coordinate" x_i so defined has mean of zero and standard deviation of one. Upon substitution into the failure function, a new failure function $Z_i(\underset{\sim}{x})$ is defined in terms of these reduced variables $\underset{\sim}{x}$. The Hasofer-Lind (H-L) generalized safety index is defined as the minimum distance from the origin of the reduced coordinates to $Z_1(\underset{\sim}{x})$ the failure function in these reduced coordinates. So defined, the generalized H-L safety index gives the same value of β as the case where the limit state is linear and the variables are normal. In other cases, the H-L index, β, will differ from the Cornell index. In summary, the H-L generalized safety index provides a measure of reliability which is invariant to the mechanical formulation of the failure function, . . . and gives the same value as the Cornell index special case of the normal format.

The concept of the generalized safety index may play a key role in the PSAM project. An estimate of the probability of failure can be made by employing Eq. 9 above with the H-L index. Even though no distributional information is used in the H-L index, probability of failure so defined will provide a reasonable estimate to the actual probability of failure in many cases. More generally, Eq. 9 can be used to construct a distribution function of a response.

Relative to the PSAM program, it is important to note that estimates of the cumulative distribution function (cdf) of functions of random variables can be made employing β. Let U be a function of several design factors

$$U = f(X_i) \tag{13}$$

The cdf of U is defined as

$$F_U(u) = P(U \leq u) \tag{14}$$

So that by analogy, the failure function is $Z = U - u$. To construct the cdf, several values of u must be chosen, and the computation for β repeated, but the computations are rapid by digital computer. Wu and Wirsching have demonstrated this technique on a low cycle fatigue problem [W6].

6.4 Advanced Reliability Methods; Rackwitz-Fiessler and Chen-Lind

A principal limitation of the H-L approach is that distributional information, even if available, is not used in the analysis. In 1978 Rackwitz and Fiessler suggested a method which extends the Hasofer-Lind safety index concept to accomodate distributional information of the design factors [R1]. Their method transforms non-normal distributions into "equivalent" normal distributions by adjusting the mean and standard deviation so that the distribution and density functions of the non-normal variables and the equivalent normal variables are equal at the design point. This scheme, an iterative algorithm which converges to a safety index, is described in references E2, L5, R3, S1, T1, and W7. It has been demonstrated by Wu that probability of failure using the Rackwitz-Fiessler β in Eq. 9 produces, in many cases, surprisingly good estimates of the probability of failure [W9]. Using a digital computer, the calculations are very fast and efficient. Again, the R-F method can be employed to construct distribution functions of responses in complicated problems, and therefore may be very useful in certain aspects of the PSAM. All of these schemes are referred to as "fast probability integration methods" because they provide a very fast approximation to Eq. 2.

An extension of the R-F scheme was proposed by Chen and Lind [C1]. This method uses a three parameter equivalent normal distribution. It was anticipated that this method can produce more accurate methods of p_f than does R-F, but Wu has shown this to be not always true [W9]. In fact, Wu has developed another advanced method, which employs techniques of Rackwitz and Fiessler and Chen and Lind to produce a safety index, and extimates of probability of failure with significantly less error than R-F or C-L.

6.5 Comments on Methods Which Rely on Higher Order Approximations to the Limit State

The procedures described above are referred to as "first order methods" because the limit state is approximated as a straight line at the design point. But other advanced methods have been proposed. In general, reliability analysis can be performed by transforming the basic variables, X_i, to standard normal variables, x_i, as suggested by Rosenblatt [R5]

$$x = \Phi^{-1}[F(X)] \tag{15}$$

the inverse transformation is

$$X = F_X^{-1}\ [\Phi(x)] \tag{16}$$

The inverse transformation is substituted into the original failure function, Z, so that the transformed failure function, Z_1, can be formulated and the safety index computed. Such a procedure is expected to produce more accurate values of p_f, but the inverse transformation can be extremely complicated.

Improvements to the first order method, suggested by various authors, typically employ a higher order approximation of the limit state. For example, Ditlevsen developed bounds by inscribing and circumscribing the limit state with rotational paraboloids [D4]. Horn and Price investigated the error of the linear approximation by studying an approximating hypersphere with radius corresponding to the mean curvature at the design point [H7]. To avoid the arbitrariness of the choice of a suitable approximating limit state, Fiessler et al. investigated several possible forms of the quadratic limit state [F5]. Breitung derived an asymptotic formula for the probability of failure which considers curvatures in the limit state at the design point

[B5]. Tvedt derived two approximation formulas, both which model the failure surface with a parabolic surface at the design point, which also give an accurate estimate of p_f [T3].

Because a more accurate description of the limit state is used, these second order methods have the promise of consistantly producing better estimates of the probability of failure relative to first order methods. However, these schemes are are more complicated because the formulations are made on the transformed space and require second partial derivatives of the transformed limit state function. In the literature, only a few simple examples have been presented, e.g., Fiessler et al. [F5], Breitung [B5], Tvedt [T3]. The evidence is not entirely convincing that quadratic methods could produce consistantly accurate results relative to other methods.

In summary, it is not obvious at this time that the much more complicated second order methods will be helpful in the PSAM project. Wu has shown that his first order method works extremely well, and errors in computing probabilities are almost always acceptably small [W9].

6.6 On the Drawing Board

While it is widely recognized that higher order forms for approximating the limit state may produce more accurate estimates of probabilities, they are typically far more complex than linear forms. Wu has developed a linear limit state algorithm, an extension of the R-F and C-L methods, which is efficient and accurate [W9]. He also has under development at this time an advanced version which, based on a few check cases, seems to be faster and more accurate.

6.7 Reliability Analysis When the Limit State Function Does Not Have a Closed Form Expression

Reliability methods described above rely on a closed form expression of the limit state, e.g., Eq. 5. But there are cases where the relationship between the variables is defined only by a computer algorithm, e.g., local strain and fracture mechanics fatigue life prediction, finite element stress analysis, etc. Wu and Wirsching have presented a method whereby the computer program is run using various combinations of parameter values, and a polynomial approximating the limit state is fitted to the responses [W6]. A fast probability integration method is then employed to estimate probabilities. The good news, . . . no modification of the program which characterizes physical behavior is necessary. The bad news, . . . particularly with regard to the PSAM project, . . . a separate analysis must be done on each response variable.

7. APPLICATION OF PROBABILISTIC DESIGN THEORY TO DESIGN CODE DEVELOPEMNT

Probably the largest effort in the U.S. to implement a reliability based design criteria was the LRFD (Load and Resistance Factor Design) program to revise the AISC specifications. Work on the development of the new AISC specifications started in 1969, and it was conducted by M. K. Ravindra and T. V. Galambos. A comprehensive summary of the theoretical development is presented in eight papers in the ASCE Journal of the Structural Division, Sept. 1978 [G2]; historical summary is provided by Galambos [G3]. The proposed specifications [P4] are now open for public review and discussion. After final revision the new rules will be included as an alternate to the 1978 AISC specifications. The lognormal format, employed in LRFD, may be useful for elementary reliability analyses in the PSAM project. In particular, a closed form expression for the probability distribution of response is possible when the random variables (all assumed to be lognormal) can be factored outside of the stiffness and mass matrices in the static linear case. Furthermore, the partial safety factor format of LRFD can be employed if a safety check expression is required.

Other well documented efforts to develop probability based design requirements are available. An excellent description of the review and revision of the National Building Code of Canada is provided by Siu, Parimi, and Lind [S3]. The NBS report by Ellingwood et al. recommends load factors and load combinations compatible with loads in the proposed 1980 version of American National Standard A58 [E2]. Load factors were developed using concepts of probabilistic limit states design. Both the Ellingwood report and the CIRIA 63 report [R3] provide excellent and comprehensive summaries of techniques and applications of modern probabilistic design theory. Moreover, Ellingwood, et al. [E2] and Galambos and Ravindra [G1] provide useful data summaries on material behavior (structural steel at room temperature). The purpose of the CIRIA 63 report was to review suitable methods for the determination of partial factors for use in limit state structural codes. Both reports detail the definition and process for computing the generalized safety index, a technique which may be very useful in all aspects of the PSAM project. Another very excellent reference is the Bulletin d'Information 112, published by the Comité Européen du Béton [R2]. Unfortunately, this document is not readily available, but it does contain major contributions from many of the pioneers of the development of probabilistic design theory.

8. A NOTE ON MONTE CARLO METHODS

Monte Carlo is employed very effectively to analyze complicated problems in probability theory, mathematical statistics, reliability, random process theory, etc. As a general rule, Monte Carlo analysis tends to be very costly relative to the accuracy of the results. Therefore, it is commonly used in a research role to verify the performance of more efficient numerical methods. In the PSAM project it is not likely to be an effective design tool.

Monte Carlo is particularly inefficient for mechanical reliability problems because accurate estimates of the small probabilities of failure require very large sample sizes. Efficiency can be improved by discrimination in sampling or by extrapolating an empirical distribution function; but generally speaking, advanced reliability methods cited in Section 6 are far more efficient for the basic reliability problem.

Monte Carlo seems more of an art than a science, and no complete work, for engineering application, seems to exist. Elementary concepts (how to sample from various distributions) are presented in Hahn and Shapiro [H1]. Both Ang and Tang [A3] and Elishakoff [E1] have chapters on Monte Carlo presenting engineering applications. Thousands of papers have been published which describe a wide variety of applications. For example, two elementary works, by this author, describe application to random process simulation for fatigue analysis [W4], and analysis of peak responses to nonstationary random forces [W3].

LIST OF REFERENCES

A1. Ang, A. H.-S., and Cornell, C. A., "Reliability Bases of Structural Safety and Design," Journal of the Structural Div., ASCE, Vol. 100, Sept., 1974

A2. Ang, A. H.-S., and Tang, W., Probability Concepts in Engineering Planning and Design, Wiley, 1975.

A3. Ang, A. H.-S., and Tang, W. H., Probability Concepts in Engineering Planning and Design, Vol. II, Wiley, 1984.

B1. Basler, E., "Analysis of Structural Safety," Proceedings, ASCE Annual Convention, Boston, 1960.

B2. Bendat, J. S., and Piersol, A. G., Engineering Application of Correlation and Spectral Analysis, Wiley Interscience, 1980.

B3. Benjamin, J. R., and Cornell, C. A., Probability, Statistics and Decision for Civil Engineers, McGraw-Hill, 1970.

B4. Bowker, A. H., and Lieberman, G. J., Engineering Statistics, Prentice-Hall, 1972.

B5. Breitung, K., "Asymptotic Approximations for Multi-Normal Domain and Surface Integrals," Fourth International Conference on Applications of Statistics and Probability in Soil and Structural Engineering. Universita di Firenze, Pitagora Editrice, 1983.

C1. Chen, X., and Lind, N. C., "A New Method of Fast Probability Integration," Solid Mechanics Division, University of Waterloo, Canada. Paper No. 171, June, 1982.

C2. Cornell, C. A., "Bounds on the Reliability of Structural Systems," Journal of the Structural Division, ASCE, Vol. 93, No. ST., Feb. 1967.

C3. Cornell, C. A., "A Probability-Based Structural Code," Journal of the American Concrete Institute, Vol. 66, No. 12, Dec. 1969.

D1. Dialog, Second International Workshop on Code Formats, Mexico City, Danmarks Ingeniorakademi, Building 373, 2800 Lyngby, Denmark, Jan. 1976.

D2. Ditlevsen, O., "Structural Reliability and the Invariance Problem," Rpt. No. 22, Solid Mechanics Division, U. of Waterloo, Waterloo, Ontario, Canada, 1973.

D3. Ditlevsen, O., "Evaluation of the Effect on Structural Reliability of Slight Deviations from Hyperplane Limit State Surfaces," Proc., 2nd Int'l. Wrkshp. on Code Formats, Mexico City, 1976.

D4. Ditlevsen, O., "Principle of Normal Tail Approximation," Journal of Engineering Mechanics Division, ASCE, Vol. 107, Dec., 1981.

E1. Elishakoff, I., Probabilistic Methods in the Theory of Structures, Wiley-Interscience, 1983.

E2. Ellingwood, B., Galambos, T. V., MacGregor, J. G., and Cornell, C. A., "Development of a Probability Based Load Criterion for American National Stand A58," NBS Special Publication 577, June, 1980

F1. Fatigue and Fracture Reliability (4 papers), Journal of the Structural Division, ASCE, Vol. 108, No. ST1, Jan. 1982.

F2. Ferry-Borges, J., and Castenheta, M., Structural Safety, Laboratorio Nacional de Engenharia, Lisbon, 1971.

F3. Fiessler, B., Neumann, H. J., and Rackwitz, R., "Quadratic Limit States in Structural Reliability," Journal of the Engineering Mechanics Division, ASCE, Vol. 105, Aug., 1979.

F4. First Order Reliability Concepts for Design Codes, Bulletin d'Information 112, Comite Européen du Béton.

F5. Forsell, C., "Economics and Buildings," Sunt Fornuft, 4, 1924.

F6. Freudenthal, A. M., "Safety of Structures," Transactions, ASCE, Vol. 112, 1947.

F7. Freudenthal, A. M., "Safety and the Probability of Structural Failure," Transactions, ASCE; Vol. 121, 1956.

F8. Freudenthal, A. M., "Safety Reliability and Structural Design; Journal of the Structural Division, ASCE, Vol. 77, No. 3, Mar. 1961.

F9. Freudenthal, A. M., Garrelts, J. M., and Shinozuka, M., "The Analysis of Structural Safety," Journal of the Structural Division, ASCE, Vol. 92, No. ST1, Feb., 1966.

F10. Freund, J. E., Mathematical Statistics, Prentice-Hall, 1971.

G1. Galambos, T. V., and Ravindra, M. K., "Properties of Steel for Use in LRFD," Journal of the Structural Division, ASCE, Vol. 104, No. ST9, Sept. 1978.

G2. Galambos, T. V., Ravindra, M. K., et al., . . . a series of eight papers on load and Resistance Factor Design (LRFD), Journal of the Structural Division, Vol. 104, No. ST9, Sept. 1978

G3. Galambos, T. V., "The AISC-LRFD Specification-Conception to Adoption," Probabilistic Mechanics and Structural Reliability, ASCE, 1984.

G4. Gumbel, E. J., Statistics of Extremes, Columbia Press, 1958.

H1. Hahn, G. J., and Shapiro, S. S., Statistical Models in Engineering, John Wiley, 1968.

H2. Hasofer, A. M., and Lind, N. C., "Exact and Invariant Second-Moment Code Format," Journal of the Engineering Mechanics Division, ASCE, Vol. 100, No. EM1, Feb., 1974.

H3. Haugen, E. G., Probabilistic Approaches to Design, John Wiley, 1968.

H4. Haugen, E. G., Probabilistic Mechanical Design, Wiley Interscience, 1980.

H5. Hines, W. W., and Montgomery, D. C., Probability and Statistics in Engineering and Management Science, Wiley, 1980.

H6. Hohenbichler, M., Rackwitz, R., "Non-normal Dependent Vectors in Structural Safety," Journal of the Engineering Mechanics Division, ASCE, Vol. 100, No. EM6, Dec., 1981.

H7. Horne, R., and Price, P. H., "Commentary on the Level-II-Procedure," Rationalization of Safety and Serviceability Factors in Structural Codes, CIRIA Report 63, Construction Industry Research and Information Association, London, 1977.

J1. Johnson, A. I., Strength, Safety, and Economical Dimensions of Structures, Statens Kommitte for Byggnadsforskning Meddelanded, No. 22, Stockholm, 1953.

K1. Kapur, K. C., and Lamberson, L. R., Reliability in Engineering Design, Wiley, 1977.

K2. Kecegioglu, D. B., and Cormier, D., "Designing a Specified Reliability into a Component," Proceedings of the Third Reliability and Maintainability Conference, Washington, D. C., 1964.

K3. Kjerengtroen, L., Reliability Analysis of Series Structural Systems, Ph.D Dissertation, The University of Arizona, 1985.

L1. Leporati, E., The Assessment of Structural Safety, Research Studies Press, 1979.

L2. Lind, N. C., Turkstra, C. J., and Wright, D. T., "Safety, Economy, and Rationality of Structural Design," Proceedings IABSE 7th Congress Rio de Janeiro, 1965.

L3. Lind, N. C., "The Design of Structural Design Norms," Journal of Structural Mechanics, Vol. 1, No. 3, 1973.

L4. Lipson, C., and Sheth, N. J., Statistical Design and Analysis of Engineering Experiments, McGraw-Hill, 1973.

L5. Lind, N. C, Krenk, S., and Madsen, H. O., Safety of Structures, to be published by Prentice-Hall, 1985.

L6. Lindgren, Statistical Theory, Macmillan, 1962.

M1. Mann, N. R., Schafer, R. E., and Singpurwalla, N. D., Methods for Statistical Analysis of Reliability and Life Data, Wiley, 1974.

M2. Mayer, M., Die Sicherheit der Bauwerke, Springer-Verlag, Berlin, 1926.

M3. Meyer, P. L., Introductory Probability and Statistical Applications, Addison-Wesley, 1970.

M4. Mood, A. M., and Graybill, F. A., Introduction to the Theory of Statistics, McGraw-Hill, 1963.

M5. Moses, F., and Stevenson, J. D., "Reliability-based Structural Design," Journal of the Structural Division, ASCE, Vol. 96, Feb. 1970.

P1. Probability Mechanics and Structural Reliability, ASCE, edited by A. H. S. Ang and M. Shinozuka, 1979.

P2. Probabilistic Methods in Structural Engineering, ASCE, edited by M. Shinozuka and J. T. P. Yao, 1981.

P3. Probability Mechanics and Structural Reliability, ASCE, edited by Y. K. Wen, 1984.

P4. Proposed Load and Resistance Factor Design Specificaiton for Structural Steel Buildings, AISC, 1983.

P5. Pugsley, A., The Safety of Structures, Edward Arnold, 1966.

P6. Pugsley, A., "Concept of Safety in STructural Engineering," Proceedings, Ins. of Civil Engineers, 1951.

R1. Rackwitz, R., and Fiessler, B., "Structural Reliability Under Combined Random Load Sequences," Journal of Computers and Structures, Vol. 9, 1978.

R2. Rackwitz, R., "Practical Probabilistic Approach to Design," First Order Reliability Concepts for Design Codes, Comité Européen du Béton, Bulletin d'Information, No. 112, July, 1976.

R3. Rationalisation of Safety and Serviceability Factors in Structural Codes, Report 63, CIRIA, Construction Industry Research and Information Association, 6 Storey's Gate, London SWIP 3AU, July, 1977.

R4. Ravindra, M. K., and Galambos, T. V., "Load and Resistance Factor Design for Steel," Journal of the Structural Division, ASCE, Vol. 104, No. ST9, Sept., 1978.

R5. Rosenblatt, M., "Remarks on a Multivariate Transformation," Annals of Mathematical Statistics, Vol. 23, No. 3, Sept., 1952.

R6. Rosenblueth, E., and Esteva, L., "Reliability Bases for Some Mexican Codes," Probabilistic Design of Reinforced Concrete Buildings, Publication SP-31, American Concrete Institute, Detroit, Mich., 1972.

S1. Shinozuka, M., "Basic Analysis of Structural Safety," Journal of the Structural Division, ASCE, Vol. 109, No. 3, March, 1983.

S2. Siddall, J. N., Probabilistic Engineering Design, Dekker, 1983.

S3. Siu, W. W. C., Parimi, S. R., and Lind, N. C., "Practical Approach to Code Calibration," Journal of the Structural Division, ASCE, Vol. 101, No. ST7, July, 1975.

S4. "Structural Safety - A Literature Review," Journal of the Structural Division, ASCE, Vol. 98, No. ST4, April, 1972.

S5. Structural Safety (6 papers), Journal of the Structural Division, ASCE, Vol. 100, No. ST9, Sept., 1974.

T1. Thoft-Christensen, P., Baker, M. J., Structural Reliability Theory and Its Applications, Springer-Verlag, N. Y., 1982.

T2. Turkstra, C. J., Theory of Structural Safety, SM No. 2, Solid Mechanics Division, U. of Waterloo, Waterloo, Ontario, Canada, 1970.

T3. Tvedt, L., "Two Second Order Approximations to the Failure Probability," Det Norske Veritas (Norway) RDIV/20-004-83, 1983.

W1. Weibull, W., "A Statistical Theory of the Strength of Materials," Proceedings, Royal Swedish Institute of Engineering Research, No. 151, Stockholm, 1939.

W2. Weibull, W., "A Statistical Distribution Function of Wide Applicability," Journal of Applied Mechanics, ASME, Vol. 18, 1951.

W3. Wirsching, P. H., and Yao, J. T. P., "Monte Carlo Study of Seismic Structural Safety," Journal of the Structural Division, ASCE, Vol. 97, No. ST5, May, 1971.

W4. Wirsching, P. H., and Light, M. C., "Fatigue Under Wide Band Random Stresses," Journal of the Structural Division, ASCE, Vol. 106, No. ST 7, July, 1980.

W5. Wirsching, P. H., "Application of Probabilistic Design Theory to High Temperature Low Cycle Fatigue," NASA CR-165488, NASA Lewis Research Center, Cleveland, OH., Nov., 1981.

W6. Wu, Y.-T., and Wirsching, P. H., "Advanced Reliability Method for Fatigue Analysis, Journal of Engineering Mechanics, ASCE, Vol. 110, No. ST4, April, 1984.

W7. Wirsching, P. H., and Wu, Y.-T., "Reliability Considerations for the Total Strain Range Version of Strainrange Partitioning," NASA CR 174757, NASA/Lewis RC, Cleveland, OH., Sept., 1984.

W8. Wirsching, P. H., and Wu, Y.-T., "A Review of Modern Approaches to Fatigue Reliability Analysis and Design," ASME Fourth National Congress on Pressure Vessel and Piping Tehcnology, Portland, OR, June, and published in Random Fatigue Life Prediction, ASME, 1983.

W9. Wu, Y.-T., Efficient Algorithm for Performing Fatigue Reliability Analyses, Ph.D. Dissertation, The University of Arizona, 1984.

Section 2

Literature Review on Probabilistic Structural Analysis and Stochastic Finite Element Methods

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ABSTRACT

The notion of stochastic variables in structural analysis was introduced by the late Professor A.M. Freudenthal as early as in 1945. The goal has been to assess structural safety in a rational fashion. One cannot totally rely on the hypothetical deterministic assumptions with the pretention that the knowledge is complete and exact regarding material properties, geometry of components, and loading. Hence the Probabilistic Structural Analysis Method (PSAM) emerged in order to evaluate structural performance in real world situations. Along with the advent of digital computers the finite element method has established itself to be the singlemost versatile numerical tool for engineering calculation. Stochastic analysis on the response database furnished by a finite element scheme is then the most logical way to carry out relevant reliability calculations for engineers who are responsible to assure safe functionality of systems they analyze, design and construct.

Quantitative estimation of failure apprehension can be obtained by considering stochasticity of both loading and structural description. The former aspect is treated in random vibration and will not be addressed here. Available finite element type formulations with random variables describing stiffness, mass and damping matrices due to uncertainities in boundary geometry, initial stress distributions,. material properties and bounday conditions, are reviewed in this report.

Computational procedure for evaluating the design statistics (such as the means, variations, correlations, etc.) of mode shapes, resonant frequencies, buckling loads and non-linear dynamic respnses are summarized. A list of reference of important publications is furnished. Comments on outstanding issues and necessary research is also included herein.

1. Introduction

Engineering systems are designed with a variety of materials and are shaped conveniently in order to perform certain functions. During its service life a system enounters many different static and dynamic loading conditions. The main concern that spans from a lay person to a competent designer is (a) whether the structure will survive, (b) how well the behavior of the structure would correspond to the required specifications and, (c) what are the chances of encountering undesirable circumstances such as cracks and excessive vibrations. Everyone is interested in the the overall rating of performance as well. We can immediately detect that the direction of these natural questions are both quantitative as well as qualitative in nature. If we consider the entire design procedure to be a decision making activity, then at each instance we are compelled to resolve a generic question. What is the chance that certain criterion will not be met during the life of the engineering system which is conceived on a design board?

We immediately recognize that the problem in engineering design analysis is bifocal. First, we must recognize physical behaviors and secondly, we must examine the extent of our knowledge regarding these behaviors. In order to answer the first question, we axiomatize a mathematical model and quantify applicable physical laws. Then analysis is performed adhering as closely as possible to exact solutions. Unfortunately, even many timple objects of engineering design analysis are so complex in geometry that continuum methods succeeded by analytic solutions reduces to nothing more than text book examples. Thus in practice, based on the knowledge of systems of rather simplified geometry, discrete (as opposed to continuum) methodologies are pursued where the solutions are arrived at in numerial steps (contrary to analytical methods with closed form expressions). Computational methods such as finite difference and finite element techniques thus emerged as very powerful numerical tools. With the advent of high speed digital computers, it became possible to carry out a large number of arithmetic operations leading to the success of those numerical methods appropriate for dynamic response computation as well as thermal analysis. Thus the partial differential equations of mathematical physics, which dictate the motion, thermal behavior, etc., are reduced to rather simplified solution of algrebaic equations. The finite element method, which is a means to spatially discretize the continuum operator that governs the field variables, became very popular since the material inhomogeneity, anisotropy, arbitrariness of boundary geometry could be easily incorporated in the numerical procedures. In essence, the answer to the first question can be summarized in terms of applicability of the conventional finite element method.

However, the second question invokes a different branch of discipline altogether viz. probabilistic analysis and statistical computations. We have first hand experience that the design assumptions are quite empirical if not gross to some extent. In reality we are dealing with partial, often quite incomplete and contaminated information regarding the structure and loading conditions. Hence it is quite legitimate to attempt to evaluate differences between the predicted and any possible realistic responses. Very naturally, concepts like mean values, standard deviation, probability distributions and exceedence (probability to exceed the allowable limits) arise within the selected numerical method i.e., the finite element method. Thus a conjugation of the finite element procedure (spatial descretization) with the probabilistic notion of analysis becomes ineviable in a rational design-analysis environment.

In order to illustrate the aforementioned generalized (to some extent rather vague) discussion let us consider one of the most simple problems in structural mechanics. This will also facilitate the introduction of some definitions like random variables, stochiastic processes, etc. which are vital to the appreciation of the cited literature reviewed in the succeeding chapters.

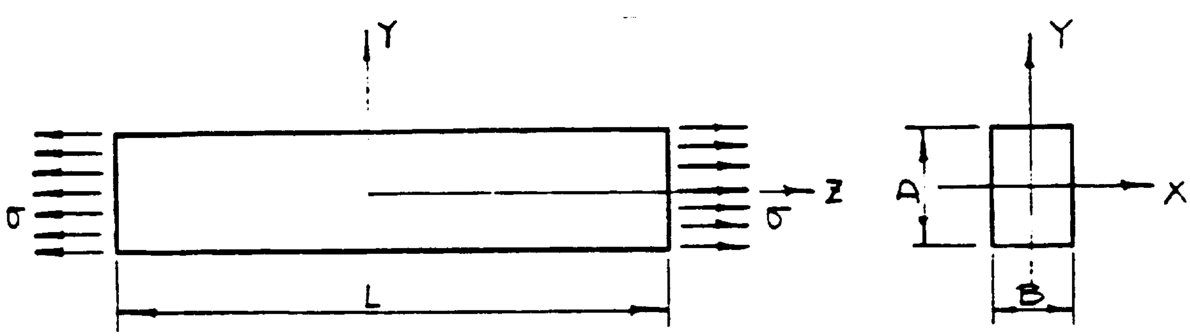

Fig.1.1 Uniaxial Bar Problem

Consider in Fig. 1.1 a uniform bar of length L, depth D, width B, subjected to a constant axial stress σ. The material will be taken to be homogenous thus the modulus of elasticity E will be considered to be constant. Suppose we are interested in the strain ε and elongation U of the member. From basic stength of materials:

$$\varepsilon = \frac{\sigma}{E} \quad \text{and} \quad U = \varepsilon L = \frac{\sigma L}{E} \tag{1.1}$$

Now we shall ask a pertinent question regarding our confidence in the assumptions leading to the expressions of ε and U in equation (1.1). The first set of questions will address the loading. How accurately do we know that σ is uniform on the end surfaces? If there is a device which applies the force we can never be sure that a perfectly uniform stress condition is imposed. The rational way to proceed will be to estimate functions $F_{\sigma}(x,y,\sigma)$ on the left and right faces such that at a point (x,y) the probability of the applied stress to be less than σ will be given by the value of the function F. At this stage let us assume that the bar is "perfect" with its stipulated geometrical dimensions and modulus of elasticity. The resulting strain distribution $\varepsilon(x,y,z)$ will also now become uncertain as a consequence of the distribution $F_{\sigma}(x,y,\sigma)$. Then the pertinent design quantity to look for, in order to perform an analysis on the basis of strain, will be $F_{\varepsilon}(x,y,z,\varepsilon)$, i.e., the probability distribution function for the strain ε. It is interesting to note that the stochastical strain now becomes a three-dimensional function even for the corresponding one-dimensional deterministic case. Hence we need to carry out a three-dimensional analysis of the aforementioned bar of Fig. 1.1. In order to utilize an available finite element computer program we spatially discretize this static problem. Without any loss in generality and especially in order to avoid unnecessary complexity let us assume that the end stresses are so applied that σ does not vary with x. Hence the probability distribution function for σ could be represented in the form $F_\sigma(y, \sigma)$. If from our engineering insight we assume that the resulting strain ε does not vary with x at a certain section then we would like to evaluate $F_\varepsilon(y, z, \varepsilon)$. For this two-dimensional idealization we employ a two-dimensional finite element mesh as shown in Fig. 1.2.

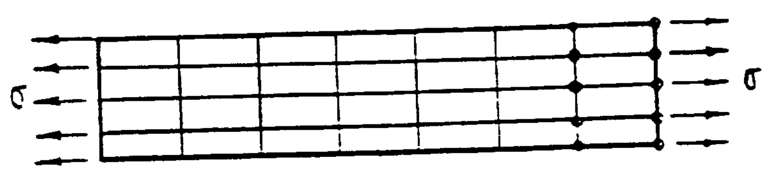

Fig.1.2 Finite Element Discretization

The objective of the probabilistic finite element procedure within the context of the problem in Figs. 1.1 and 1.2 is to evaluate $F_{\varepsilon}(y,z,\varepsilon)$ when $F_{\sigma}(y,\sigma)$ are given for each end faces.

In the preceeding example we consider the stress, σ, loading (forcing function in the finite element system) to be a nondeterministic function of x and y. This will be termed to be a stochastic or random process. A formal definition of a random process is that f(x) is a random process if f is a random function of a deteministic argument x. We shall indicate stochastical variables with a tilda. Thus a random function such as the stress, strain in the above problem will be random processes, $\tilde{\sigma}(y)$ and $\tilde{\varepsilon}(y,z)$, respectively. (Notation introduced in [B-1.4] will be used throughout this report.)

A dynamic version of the above class of problems, depicted in Figs. 1.1 and 1.2, attracted the notice of several researchers. Therein a structure or any other mechanical system was considered to be completely deterministic whereas only the forcing function (such as the earthquake or wind load) was considered to be random processes in time. This class of problem of deterministic system with random loading are treated in a special branch of dyamics called random vibration, refer to [C-1.4]. There are excellent standard text books on that topic as listed in the reference (section 10).

We can further pursue our question regarding the assumptions in the formulation leading to $\varepsilon = \frac{\sigma}{E}$, $U = \frac{\sigma L}{E}$ in equation (1.1). There are possibilities that during manufacturing of the "real world" bar in Fig. 1 the chemical process was not exact or perfect hence the modulus of elasticity E may just be almost constant and is indeed a random function of x,y.and z, i.e. $\tilde{E}(x,y,z)$. Since the manufacturing processes are quite reliable we expect $\tilde{E}$ to be a narrow band process implying that the difference in say the maximum and minimum values of E everywhere will not be too large. Statistically speaking the dispersion in E will not be enormous. Similarly a realistic (non ideal) manufacturing proces will incur variation in the depth D and width B of the bar (refer to Fig. 1.1). Thus B and D are to be taken as stochastic processes: $\tilde{B}(z,y)$ and $\tilde{D}(z,x)$, respectively. Most likely the departures ΔB and ΔD of the width and depth from corresponding mean values B_o and D_o [$\tilde{B} = B_o + \Delta B$ similarly $\tilde{D} = D_o + \Delta D$] will be confined within a few percentage points. Now the estimation of probability distribution function $F_{\tilde{\varepsilon}}$ in terms of $F_{\tilde{\sigma}}$, $F_{\tilde{B}}$, $F_{\tilde{D}}$ and $F_{\tilde{E}}$ will not be a simple algebraic task. In fact the deterministic equations (1.1) may not even be valid for the mean values, i.e.

$$\varepsilon_o \overset{?}{\neq} \frac{\sigma_o}{E_o} \qquad (1.2)$$

The stochasticity in material properties and in geometry which modify the system stiffness is of principal importance here in the estimatior of randomness for strains, displacements, and any relevant response quanitity. Published papers, which deal with the estimation of mean and standard deviations (correlation matrix in the case of correlated stochastic variables) are reviewed in this report. The finite element methodology has been the focus in recommending practical solution strategies which consider randomness and particularly the spatial variability of stochastic processes in structural systems.

2. Problem Statement

We express a finite element form for the equation of motion of a structure in the symbolic form:

$$S\ R = F \tag{2.1}$$

where: S: system stiffness operator

R: system response history

F: forcing function

For a vector of random processes $\tilde{X}$ which define the system S, the corresponding stochastical finite element system of equation will then be

$$\tilde{S}\ \tilde{R} = \tilde{F} \tag{2.2}$$

In a generalized probabilistic finite element problem we shall have:

$F_{\tilde{X}}$: the probability distribution function for basic uncertain quantities are prescribed.

$F_{\tilde{R}}$: the probability distribution functions for response quantities are to be calculated.

From a design-analysis view point the probability that a certain response quantity R_i would exceed a predetermined prescribed value R_i^*, i.e.

$F_{\tilde{R}_i}(R_i^*)$: Exceedence of R_i with respect to R_i^*

is also very important.

The key issues are then:

i) Construction of the system stochastic finite element matrices to decribe $\tilde{S}$ in (2.2)

ii) Solution of $\tilde{R}$ from (2.2)

iii) Evaluation of the probability distribution function $F_{\tilde{R}}$ from the solution.

In this report the effect of randomness for responses (say computation of $F_{\tilde{R}}$) due to system stochasticity will be highlighted. Thus for the majority of the problems reviewed herein we shall specialize the fully probabilistic equation (2.2) with deterministic forces:

$$\tilde{S}\,\tilde{R} = F_o \tag{2.3}$$

It may be remarked that the effects of random loading with deterministic systems, such that:

$$S_o\,\tilde{R} = \tilde{F} \tag{2.4}$$

can be found in various publications on random vibrations. Research with system stochasticity refer to equation (2.3) are rather scarce compared to quite a rich literature available on random excitation. From the practical consideration in the structural engineering application it is adequate to postulate narrow band processes for a stochastic quantity $\tilde{X}_i$. In some sense of a norm $\|.\|$ one can than write:

$$\frac{\| \Delta x_i \|}{\| x_i \|} << 1 \qquad (2.5)$$

[A norm $\|.\|$ is a quantification where $\|\tilde{X}_i\|$ is a real positive (non-negative) number associated with a physical variable $\tilde{X}_i$] Consequently, it is quite pertinent to propose that the system components and the responses as well will obey inequalitites similar to equation (2.5). This naturally makes the pertubation method very attractive in analyzing the system equation (2.3). From its incipience the stochastical finite element method resorted to perturbation expansion in forms of Taylor series about mean inputs in order to yield required statistics related to $F_{\tilde{R}}$.

In the sections that follows noteworthy papers which deal with system stochasticity, refer to equation (2.3), will be summarized with brief description of solution procedures and published numerical results.

3. Papers of Historical Importance

Boyce in 1961, published a paper of column buckling with stochastical initial displacement. ("Buckling of a Column with Random Initial Displacement", Journal of Aero. Sci.). The eigenvalue problem related to free vibration of structures with stochastical mass and stiffness matrices was completed in 1968 by Collins and Thomson ("The Eigenvalue Problem for Structural Systems with Stochastical Properties", AIAA Journal). The pertubation method introduced by Collins and Thomson was later adopted by many researchers, such as Nakagiri and Hisada [N-4.1 to N-4.8]. There latter authors derived the mass and stiffness matrices by employing the finite element method. The aforementioned two papers and [C-3.1] [B-3.1] are of historical significance in the research of structural mechanics problems with system stochasticity.

From the standpoint of Probabilistic Structural Analysis Method (PSAM) the first paper the reviewer found of direct interest is by Mak, and Kelsey, in 1971 titled: "Statistical Aspects in the Analysis of Structures with Random Imperfunctions". Cambou in 1972, employed a direct finite element formulation in first order stochastic analysis for linear elasticity problems [C-3.2].

Mak and Kelsey [M-3.1] considered the out-of-plane buckling bifurcation of a column due to uncertainty in the initial stress distribution. This is the first published mathematical development with numerical results for any structural problem with probabilistic consideration for the system stiffness matrix. The authors solved the eigenvalue problem asociated with the buckling problem:

$$[K + \frac{\tilde{S}}{L} K_e + \frac{\tilde{P}}{2L} K_g] \tilde{U} = 0 \qquad (3.1)$$

where K: elastic stiffness matrix

K_e: initial stress effect

K_g: geometrical stiffness matrix

and $\tilde{S}$ and $\tilde{P}$ are the stochastical initial force and the resulting buckling load. A similar development was adapted by Nakagiri and Hisada in their paper [N-4.4]. The detail of algebraic steps are furnished in section 4. Mak and kelsey in [M-3.1] presented a graph showing the probability distribution of failure by buckling and the effect of the standard deviation of the lack of fit for members on the probability of failure. The treatment in the paper are very clear and structural designer will find it suitable for application in practical problems.

The Monte Carlo simulation technique with a finite element formulation was employed by Astill, Nosseir and Sinozuka as early as in 1971, refer to [A-9.1] and section 9 for detail. The authors devised a "front end" statistical package to generate a population of constitutive properties. The problem of failure of a concrete cylinder was considered under impact loading. Very encouraging results from that transient dynamic problem with one hundred realizations was reported.

4. General Procedures for Stochastic Finite Elements

In the existing literature, stochastical analysis of engineering systems are confined to the first- and second-order, second-moment approximations. The method calls for the first- and second-order Taylor expansion of any generic response quantity in terms of system random variables around the mean argument. Subsequently the mean and standard deviation of the response function in question can be estimated. This procedure is known as the delta method by statisticians. Hitherto emphasis has been placed on reliability analysis whereby the exceedance coefficients are estimated on the basis of means and dispersions of response quantities. A more precise estimation of exceedance calculation will necessitate the knowledge of higher-order moments. The existing literature on stochastic finite elements is rather deficient in evaluating these higher-order moments.

The methodology for the first-order second-moment approximation is quite complete for linear systems. The second-order perturbation formulations are rather recent. Even though the methodology is straightforward and conceptually amenable to nonlinear dynamic systems, the details of computational strategies suitable for finite element systems especially with a large number of stochastic parameters cannot be found in existing literature.

A thorough review of published research in the are of probabilistic analysis for finite element systems reveals two major directions. Theoretically, the perturbation formulation and numerically the Monte Carlo simulation are the only courses available so far. In this section stochastic finite element formulations according to the perturbation method is detailed. The notion of finite element spatial discretization in a stochastic model on the basis of the scale of fluctuation is also reviewed here. The Monte Carlo simulation technique is more of a statistical method hence it is described in the next chapter.

Perturbation Method

The systematic development of the stochastical finite element formulation according to the perturbation method was initiated by Nakagiri and Hisada, refer to [N-4.1] - [N-4.8]. They essentially employed the perturbation method [B-1.4] and reatined up to second-order terms. In order to focus on the stochasticity of the system, the load vector (the right-hand side of the equation of equilibrium) was taken to be deterministic. In this review, the equations furnished by Nakagiri and Hisada will be rewritten using the notations that appear in [B-4.1]. In the interest of clarity, indicial notation will be employed whenever required.

The general discussion may be started by examining the stochastic static (global) stiffeners matrix $\tilde{K}$ as an offset by ΔK from a preselected deterministic value K_0, then

$$\tilde{K} = K_0 + \Delta K \tag{4.1}$$

Now for each element ij, the equation reduces to:

$$\tilde{K}_{ij} = K_{0ij} + \Delta K_{ij}$$

A superscript will be used here to indicate the corresponding variable pertaining to an element "s"; hence

$$\tilde{K}_{ij}^{(s)} = \tilde{K}_{0ij}^{(s)} + \Delta K_{ij}^{(s)} \tag{4.2}$$

The random variables which govern the system stochasticity, are collected as a vector {X} with components X_i. Conceptually, both the global and element stiffness matrices, K and $K^{(s)}$, respectively, can be Taylor expanded about a preselected vector X_0 where

$$\tilde{X}_i = X_{0i} + \Delta X_i \tag{4.3}$$

leading to

$$\tilde{K}_{ij} = K_{0ij} + \Delta K_{ij} = K_{0ij} + \frac{\partial \tilde{K}_{ij}}{\partial \tilde{X}_\ell} \Delta X_\ell + \frac{1}{2!} \frac{\partial^2 K_{ij}}{\partial \tilde{X}_\ell \partial \tilde{X}_m} \Delta X_\ell \Delta X_m \tag{4.4}$$

or

$$\tilde{K}_{ij} = K_{-ij} + \alpha_{ij\ell} \Delta X_\ell + \beta_{ij\ell m} \Delta X_\ell \Delta X_m \tag{4.5}$$

where

$$\alpha_{oj\ell} = \frac{\partial K_{ij}}{\partial \tilde{X}_\ell} \quad \text{and} \quad \beta_{ij\ell m} = \frac{\partial \tilde{K}_{ij}}{\partial \tilde{X}_m \partial \tilde{X}_\ell}$$

It should be noted that in the above equation, the expressions beyond the quadratic terms are truncated by Nakagiri and Hisada. Ther is no such restriction (refer to Eq. 3 in [B-1.4]) in a general perturbation technique. A similar expansion, consistent with the second-order perturbation of the stiffness matrix, can be implemented for the displacement vector U:

$$\tilde{U}_i = U_{0i} + \frac{\partial \tilde{U}_i}{\partial \tilde{X}_j} \Delta X_j + \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_j \partial \tilde{X}_\ell} \Delta X_j \Delta X_\ell \tag{4.6}$$

Now a static finite element system, with a deterministic load F_0 can be solved when the stiffness matrix and consequently the displacement vector are stochastical in nature. The governing equation of equilibrium then becomes

$$\tilde{K}_{ij}\tilde{U}_j = F_{0i} \tag{4.7}$$

Now, substitution of Eqs. 4.6 and 4.5 into the above equation leads to

$$[K_{0ij} + \alpha_{ij\ell}\Delta X_\ell + \beta_{ij\ell m}\Delta X_\ell \Delta X_m] \cdot$$

$$\cdot [U_{0j} + \frac{\partial \tilde{U}_j}{\partial \tilde{X}_\ell} \Delta X_\ell + \frac{\partial^2 \tilde{U}_j}{\partial \tilde{X}_\ell \partial \tilde{X}_m} \Delta X_\ell \Delta X_m] = F_{0i} \tag{4.8}$$

One compares the zero-th, first and second degree terms containing ΔX_ℓ, ΔX_m,
etc. and obtains the following recursive set of equations:

$$K_{0ij}U_{0j} = F_{0i} \tag{4.9a}$$

$$K_{0ij} \frac{\partial \tilde{U}_j}{\partial \tilde{X}_\ell} = - \alpha_{ij\ell}U_{0j} \tag{4.9b}$$

$$K_{0ij} \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_\ell \partial \tilde{X}_m} = - [\beta_{ij\ell m}U_{0j} + \alpha_{ijm} \frac{\partial \tilde{U}_j}{\partial \tilde{X}_\ell}] \tag{4.9c}$$

It is interesting to note that the above system of equations can be solved once the K_0 matrix is "inverted." Nakagiri and Hisada remarked that numerical calculation will be faster in their method as compared to a Monte Carlo simulation since the latter necessitates a separate inversion at each numerical realization for the random vector $\tilde{X}$.

Once U_{0i}, $\frac{\partial \tilde{U}_j}{\partial \tilde{X}_\ell}$ and $\frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_\ell \partial \tilde{X}_m}$ are obtained by solving a set of linear systems of equations (Eq. 4.9), one can compute the expected value of $\tilde{U}$. The expected value operator **E**, when applied on the Taylor expanded form for $\tilde{U}$ (Eq. 4.6), one obtains

$$\mathbf{E}[U_i] = U_{0i} + \frac{\partial \tilde{U}_i}{\partial \tilde{X}_j} \mathbf{E}[\Delta X_j] + \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_j \partial \tilde{X}_\ell} \mathbf{E}[\Delta X_j \ \Delta X_\ell] \tag{4.10}$$

The authors suggested that in a "deterministic" computation with K_0, the stochastic vector X_0 should be chosen to be the mean of $\tilde{X}$. Then

$$\mathrm{E}[\tilde{X}_i] = X_{0i} \quad \text{i.e.} \quad \mathrm{E}[\Delta X_i] = 0 \tag{4.11}$$

This would simplify the expression for the mean $\tilde{U}$ in Eq. 2.10 leading to

$$\mathbf{E}[U_i] = U_{0i} + \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_j \partial \tilde{X}_\ell} \mathbf{E}[\Delta X_j \Delta X_\ell] \tag{4.12}$$

It is convenient to introduce the covariant matrix $\mathrm{Cov}[\tilde{X},\tilde{X}]$ such that

$$\mathrm{E}[\Delta X_i \ \Delta X_j] = \mathrm{Cov}[\tilde{X},\tilde{X}]_{ij} = \mathrm{Cov}[\tilde{X}_i,\tilde{X}_j] \tag{4.13}$$

Now the mean displacement can be computed from:

$$\mathbf{E}[\tilde{U}_i] = U_{0i} + \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_j \partial \tilde{X}_\ell} \mathrm{Cov}[\tilde{X}_j,\tilde{X}_\ell] \tag{4.14}$$

The second-order Taylor expansion of $\tilde{K}$ and $\tilde{U}$ in Eqs. 4.4 and 4.6 limits up to second-moment terms in the above equation. Consistent with these second-moment terms in Eq. 4.14, one evaluates the dispersion of $\tilde{U}_i$ in terms of the variance operator $\mathrm{Var}[U_i]$:

$$\mathrm{Var}[U_i] = \mathbf{E}[U_i^2] - \{\mathbf{E}[U_i]\}^2 \tag{4.15}$$

The second term of the right-hand side in the above equation was obtained from Eq. 4.14. The first term on the right-hand side is calcualted retaining only the second-order terms leading to

$$E[U_i^2] = U_{0i}^2 + 2U_{0i} \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_j \partial \tilde{X}_\ell} \mathrm{Cov}[\tilde{X}_j, \tilde{X}_\ell]$$

$$+ \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_j} \frac{\partial^2 \tilde{U}_i}{\partial \tilde{X}_\ell} \mathrm{Cov}[\tilde{X}_j, \tilde{X}_\ell] \tag{4.16}$$

Once the expected value $E[U_i]$ and dispersion $var[U_i]$ are obtained from Eqs. 2.14 and 2.16, the corresponding statistics of the strain $\tilde{\varepsilon}$ and stress $\tilde{\tau}$ can be obtained by utilizing the strain-displacement transformation $\tilde{B}$ in terms of the shape functions $\tilde{N}$ and the constitutive tensor $\tilde{C}$. The algebra is summarized in Eqs. 15-21 in [B-1.4].

The aforementioned general technique, described by equations 4.1 through 4.16, was illustrated by Nakagiri and Hisada in their first paper [N-4.1] where only the variation of the shape functions were considered. For a triangular meshing, a shape function N was written in terms of the area coordinates L_1, L_2 and L_3 and the nodal point coordinates. This is a standard finite element procedure and the details can be obtained from [Z-1.1]. In this first paper, the stochasticity of the nodal coordinates were considered. An element stiffness matrix $K^{(s)}$ in an isoparametric formulation was obtained from the corresponding stochastic strain-displacement transformation matrix $\tilde{B}^{(s)}$, the constitutive matrix (stress-strain relationship) $C^{(s)}$ as well as the Jacobian transformation $\tilde{J}^{(s)}$ whose stochasticity is due to those of the nodal points. Integration over the element in terms of the area coordinates L_1 and L_2 yielded:

$$K^{(s)} = \iint [\tilde{B}^{(s)}]^T C^{(s)} \tilde{B}^{(s)} |\tilde{J}^{(s)}| dL_1 dL_2 \quad (4.17)$$

where the determinant of $\tilde{J}^{(s)}$ is indicated by $|\tilde{J}^{(s)}|$. For a nodal point coordinate (x,y), the Jacobian assumed the form:

$$\tilde{J} = \begin{bmatrix} \left(\frac{\partial \tilde{X}}{\partial L_1} - \frac{\partial \tilde{X}}{\partial L_3}\right) & \left(\frac{\partial \tilde{Y}}{\partial L_1} - \frac{\partial \tilde{Y}}{\partial L_3}\right) \\ \left(\frac{\partial \tilde{X}}{\partial L_2} - \frac{\partial \tilde{X}}{\partial L_3}\right) & \left(\frac{\partial \tilde{Y}}{\partial L_2} - \frac{\partial \tilde{Y}}{\partial L_3}\right) \end{bmatrix} \quad (4.18)$$

which was written as

$$\tilde{J} = J_0 + \Delta J \quad (4.19)$$

The terms in the $\tilde{B}^{(s)}$ matrix involved expressions like $\frac{\partial \tilde{N}}{\partial x}$ and $\frac{\partial \tilde{N}}{\partial y}$ which were obtained as

$$\begin{Bmatrix} \frac{\partial \tilde{N}}{\partial x} \\ \frac{\partial \tilde{N}}{\partial y} \end{Bmatrix} = [\tilde{J}^{(s)}]^{-1} \begin{Bmatrix} \frac{\partial \tilde{N}}{\partial L_1} - \frac{\partial \tilde{N}}{\partial L_3} \\ \frac{\partial \tilde{N}}{\partial L_1} - \frac{\partial \tilde{N}}{\partial L_3} \end{Bmatrix} \quad (4.20)$$

It was then possible to evaluate the $\alpha_{ij\ell}$ and $\beta_{oj\ell m}$ terms in Eq. 4.5, once the second-order Taylor expansion of $|\tilde{J}|$, $\frac{\partial N}{\partial x}$ were obtained. The authors denoted

$$|\tilde{J}| = |J_0| + D_1 + D_2 \quad (4.21)$$

where

$$D_1 = J^0_{11}\Delta J_{22} - J^0_{12}J_{12}\Delta J_{21} - J^0_{21}\Delta J_{12} + J^0_{22}\Delta J_{11} \quad (4.22a)$$

$$D_2 = \Delta J_{11}\Delta J_{22} - \Delta J_{12}\Delta J_{21} \quad (2.22b)$$

The required inversion of $\tilde{J}^{(s)}$ in Eq. 4.10 was expressed as

$$[\tilde{J}]^{-1} = \left(1 - \frac{D_1}{|J_0|} - \frac{D_2}{|J_0|} + \frac{D_1 D_2}{|J_0|^2}\right) \begin{bmatrix} J_{022} & -J_{012} \\ -J_{021} & J_{011} \end{bmatrix}$$

$$+ \left(1 - \frac{D_1}{|J_0|}\right. \begin{bmatrix} \Delta J_{22} & -\Delta J_{12} \\ -\Delta J_{21} & \Delta J_{11} \end{bmatrix} \tag{4.23}$$

Finally $\alpha_{ij\ell}$, $\beta_{ij\ell m}$ tensors were obtained after Taylor expansion of x and y up to second-order terms.

An example was illustrated where the nodal coordinates were taken as stochastic processes defined by a power spectrum. A homogeneous Wiener-Khintchine relation was assumed for the correlations Cov[x,x], Cov[x,y], Cov[y,y].
The autocorrelation $R(|x_i - x_j|)$for a homogeneous stochasticity was obtained in the following form:

$$R(|x_i - x_j|) = 2 \int_0^\infty s(\lambda) \cos 2\lambda |x_i - x_j| dx \tag{4.24}$$

from a given spectrum $s(\lambda)$.

The paper does not present detailed numerical results. The computational procedure for the $\alpha_{ij\ell}$ and $\beta_{ij\ell m}$ tensors are not discussed either. It should be noted that in a practical finite element formulation with stochastic variables the computation of $\alpha_{ij\ell}$ and β_{ijem} would demand substantial numerical and programming effort.

In their second paper [N-4.2] Nakagiri and Hisada considered stochasticity in the static stiffness matrix $\tilde{K}$ due to

(i) variation of constitutive properties, where $\tilde{C}[\tilde{X}]$ is considered

and

(ii) variation in boundary data.

The general methodology described before is implemented for those two cases. The paper details out plane stress/strain examples. These steps are crucial in developing a stochastic finite element code with plane elements. However, proper adaptation of the algebraic derivations to general finite element stiffness matrices (and to mass matrices as well) could lead to the formulation pertaining to three-dimensional solid and plate or shell elements. In the interest of focusing on the method the two-dimensional linear elasticity example will be sketched out here.

The stochasticity of the constitutive properties was considered first. For a plane stress/strain element the Young's modulus E and the Poisson's ratio ν were introduced as bivariate stochastical processes, in the form of $\tilde{E}(\tilde{X})$ and $\tilde{\nu}(\tilde{X})$. The random vector $\tilde{X}$ is indeed dependent upon the spatial coordinates x_1 and x_2.

The element stiffness matrix is composed of 2x2 submatrices obtained from two shape functions $N_i\ (x_\ell)$ and $N_j(x_\ell)$, refer to [Z-1.1] for details. This submatrix can be written in the following form:

$$\tilde{\xi} \begin{bmatrix} \frac{\partial N_i}{\partial X_1}\frac{\partial N_j}{\partial X_1} + \tilde{\nu}' \frac{\partial N_i}{\partial X_2}\frac{\partial N_j}{\partial X_2} & \text{Symmetric} \\ \tilde{\nu}'' \frac{\partial N_i}{\partial X_2}\frac{\partial N_j}{\partial X_1} + \tilde{\nu}' \frac{\partial N_i}{\partial X_1}\frac{\partial N_j}{\partial Y_2} & \tilde{\nu}' \frac{\partial N_i}{\partial X_1}\frac{\partial N_j}{\partial X_1} + \frac{\partial N_i}{\partial X_2}\frac{\partial N_j}{\partial X_2} \end{bmatrix} \tag{4.25}$$

The values of the stochatic variables $\tilde{\xi}$, $\tilde{\nu}'$ and $\tilde{\nu}''$ are expressed in terms of $\tilde{E}$ and $\tilde{\nu}$ for the plane stress/strain cases:

variable	plane stress	plane strain
$\tilde{\xi}$	$\frac{\tilde{E}}{1-\tilde{\nu}^2}$	$\frac{\tilde{E}\ (1-\tilde{\nu})}{(1+\tilde{\nu})\ (1-2\tilde{\nu})}$
$\tilde{\nu}'$	$\frac{1-\tilde{\nu}}{2}$	$\frac{1-2\tilde{\nu}}{2\ (1-\tilde{\nu})}$
$\tilde{\nu}''$	$\tilde{\nu}$	$\frac{\tilde{\nu}}{1-\tilde{\nu}}$

(4.26)

As before the Young's modulus and the Poisson's ratio is Taylor expanded about their means only up to the first order terms:

$$\tilde{E} = E_0 + \frac{\partial \tilde{E}}{\partial \tilde{X}_i} \Delta X_i \quad (= E_0 + \Delta E) \tag{4.27a}$$

$$\tilde{\nu} = \nu_0 + \frac{\partial \tilde{\nu}}{\partial \tilde{X}_i} \Delta X_i \tag{4.27b}$$

This leads to the following form for the element stiffness matrix, $K^{(s)}$, with second order terms:

$$\tilde{K}^{(s)} = K_0^{(s)} + \frac{\partial \tilde{K}^{(s)}}{\partial \tilde{E}^{(s)}} \Delta E^{(s)}$$

$$+ \frac{\partial \tilde{K}^{(s)}}{\partial \tilde{\nu}^{(s)}} \Delta \nu^{(s)} \tag{4.27c}$$

$$+ \frac{\partial^2 \tilde{K}^{(s)}}{\partial \tilde{E}^{(s)} \partial \tilde{\nu}^{(s)}} \Delta E^{(s)} \Delta \nu^{(s)} + \frac{1}{2} \frac{\partial^2 \tilde{K}^{(s)}}{\partial \tilde{\nu}^{(s)2}} (\Delta \nu^{(s)})^2$$

[No sum over repeated index "s"]

It is to be noted that the stiffness matrix $\tilde{K}$ is proportional to the Young's modulus $\tilde{E}$ hence $\frac{\partial^2 \tilde{K}}{\partial \tilde{E}^2}$ is zero.

The displacement vector $\tilde{U}$ when perturbed up to second order in terms of ΔE and $\Delta \nu$ leads to

$$U = U_0 + \frac{\partial \tilde{U}}{\partial \tilde{E}^{(s)}} \Delta E^{(s)} + \frac{\partial \tilde{U}}{\partial \tilde{\nu}^{(s)}} \Delta \nu^{(s)}$$

$$+ \frac{1}{2} \left[\frac{\partial^2 \tilde{U}}{\partial \tilde{E}^{(s)} \partial \tilde{E}^{(t)}} \Delta E^{(s)} \Delta E^{(t)} \right.$$

$$+ \frac{\partial^2 \tilde{U}}{\partial \tilde{E}^{(s)} \partial \tilde{\nu}^{(t)}} \Delta E^{(s)} \Delta \nu^{(t)} \tag{4.28}$$

$$\left. + \frac{\partial^2 \tilde{U}}{\partial \tilde{\nu}^{(s)} \partial \tilde{\nu}^{(t)}} \Delta \nu^{(s)} \Delta \nu^{(t)} \right]$$

The expansion for the displacement vector involves summation over all elements (as described by the superscript "s" and "t"), whereas that for $\tilde{K}^{(s)}$ pertaining to a particular element s is described by variations of $\tilde{E}$ and $\tilde{\nu}$ in that region.

In the case of a deterministic load vector F_0 the unknown partial derivatives of $\tilde{U}$ with respect to $\tilde{E}$ and $\tilde{\nu}$ can be obtained by considering terms with ΔE and $\Delta\nu$ in

$$\tilde{K}\ \tilde{U} = F_0 \tag{4.29}$$

leading to:

$$U_0 = [K_0]^{-1}\ F_0 \tag{4.30a}$$

$$\frac{\partial \tilde{U}}{\partial \tilde{E}} = -\ [K_0]^{-1}\ [\frac{\partial \tilde{K}}{\partial \tilde{E}}\ U_0] \tag{4.30b}$$

$$\frac{\partial \tilde{U}}{\partial \tilde{\nu}} = -\ [K_0]^{-1}\ [\frac{\partial \tilde{K}}{\partial \tilde{\nu}}\ U_0] \tag{4.30c}$$

$$\frac{\partial^2 \tilde{U}}{\partial \tilde{E}^2} = -\ [K_0]^{-1}\ [\frac{\partial \tilde{K}}{\partial \tilde{E}}\ \frac{\partial \tilde{U}}{\partial \tilde{E}}] \tag{4.30d}$$

$$\frac{\partial^2\ \tilde{U}}{\partial \tilde{E}\ \partial \tilde{\nu}} = -\ [K_0]^{-1}\ [\frac{\partial \tilde{K}}{\partial \tilde{E}}\ \frac{\partial \tilde{U}}{\partial \tilde{\nu}} + \frac{\partial \tilde{K}}{\partial \tilde{\nu}}\ \frac{\partial \tilde{U}}{\partial \tilde{E}}$$

$$+ \frac{\partial^2\ \tilde{K}}{\partial \tilde{E} \partial \tilde{\nu}}\ U_0] \tag{4.30e}$$

$$\frac{\partial^2 \tilde{U}}{\partial \tilde{\nu}^2} = -\ [K_0]^{-1}\ [\frac{\partial^2\ \tilde{K}}{\partial\ \tilde{\nu}^2}\ U_0 + \frac{\partial \tilde{K}}{\partial \tilde{\nu}}\ \frac{\partial \tilde{U}}{\partial \tilde{\nu}}] \tag{4.30f}$$

The authors suggest the computation of the mean and dispersion of $\tilde{U}$ from the above expressions. As claimed by the authors to be a strong point of their formulation the aforementioned equations involve the inversion of a single $[K_0]$ matrix. (In a Monte Carlo simulation each realization would demand a separate inversion of $\tilde{K}$.)

The strain-displacement transformation matrix B is deterministic hence the strain vector $\varepsilon^{(s)}$ becomes:

$$\varepsilon^{(s)} = B^{(s)}\ \tilde{U} \tag{4.31a}$$

$$= B^{(s)}\ [U_0 + \frac{\partial \tilde{U}}{\partial \tilde{E}}\ .\ .\ .\ \text{as in (4.28)}\ ] \tag{4.31b}$$

hence the mean strain $\mathbf{E}\ [\varepsilon]$ and its dispersion Var $[\varepsilon]$ can be calculated directly. Finally the stress calculation involves the stochastic constitutive matrix $\tilde{C}$:

$$\tilde{C} = \tilde{\xi} \begin{bmatrix} 1 & \tilde{\nu}'' & 0 \\ \tilde{\nu}'' & 1 & 0 \\ 0 & 0 & \nu' \end{bmatrix} \tag{4.32}$$

Employing the explicit definition of $\tilde{C}$ from (4.26) one obtains directly those partial derivatives like $\frac{\partial \tilde{C}}{\partial \tilde{E}}$ and $\frac{\partial \tilde{C}}{\partial \tilde{\nu}}$. Substitution of these quantities lead to the expression of mean and dispersion of the stress components.

The authors have not commented on the numerical implementation of mean and dispersion calculation of the stress vector $\tilde{\sigma} = \tilde{C}\ B\ \tilde{U}$ (4.33)

The rest of the paper [N-4.2] elaborates the concept of adopting a stochastical description of the boundary data. The nodal degrees-of-freedom (with the prescribed stochasticity) which pertain to the boundary were designated with a superscript ^{2} and the remaining (interior) degrees-of freedom by 1. Then the static stiffness matrix, the displacement and the load vectors could be partitioned leading to

$$\begin{bmatrix} \tilde{K}^{(11)} & \tilde{K}^{(12)} \\ \tilde{K}^{(21)} & \tilde{K}^{(22)} \end{bmatrix} \begin{Bmatrix} \tilde{U}^{(1)} \\ \tilde{U}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \tilde{F}^{(1)} \\ \tilde{F}^{(2)} \end{Bmatrix} \tag{4.34}$$

Thus the unknown displacement vector (associated with the interior nodes) $\tilde{U}^{(1)}$ becomes:

$$\tilde{U}^{(1)} = [\tilde{K}^{(11)}]^{-1} [\tilde{F}^{(1)} - \tilde{K}^{(12)} \tilde{U}^{(2)}] \tag{4.35}$$

The authors pointed out that the aforementioned equation indicated a linear relation between $\tilde{U}^{(1)}$ and $\tilde{U}^{(2)}$ hence conjectured the possibility of numerical computation of the mean and dispersion of $\tilde{U}^{(1)}$ from those of the right hand side quantities from equation (4.35). Finally relevant statistics for the strain $\tilde{\varepsilon}$ and stress $\tilde{\sigma}$ distributions could be obtained according to the equations (4.31) through (4.33).

The authors do not include specific numerical examples for this problem of stochasticity with random boundary data. As in [N-4.1] and in (4.24) a power spectral density function in the Wiener-Khintchine form was suggested to account for the spatial variability of the stochastic quantities. The authors did not elaborate on numerical computations of means and dispersions of the stress components from those of the given constitutive matrix and calculated displacement vector, refer to (4.33).

The possibility of extending the perturbation technique sketched out in [N-4.1] and [N-4.2] for <u>nonlinear</u> problems is discussed by Nakagiri and Hisada in their third note. A specific nonlinear constitutive model in the following form was explored:

$$\tilde{\sigma} = f(\tilde{\varepsilon}) = E\,\tilde{\varepsilon} - E\,\tilde{b}[n\{1 + (\frac{\tilde{\varepsilon}}{\tilde{b}})\}^{\tilde{a}} - 1] \tag{4.36}$$

A nonlinear constitutive tensor $\tilde{C}$ could then be assumed in terms of the effective strain ε and effective stress σ, which are defined to be

$$\varepsilon = \sqrt{\begin{matrix}(\varepsilon_{x_1}^2 + \varepsilon_{x_2}^2 + \varepsilon_{x_3}^2 + \frac{1}{2}\gamma_{x_1 x_2} + \\ \frac{1}{2}\gamma x_2^2 x_3 + \frac{1}{2}\gamma x_2^2 x_3)\end{matrix}} \tag{4.37a}$$

and

$$\tilde{\sigma} = f\,(\tilde{\varepsilon}) \tag{4.37b}$$

<u>In principle</u>, for a selected value of $\tilde{U}$ to be U^* the constitutive matrix $\tilde{C}^{(s)}$ for an element "s" was Taylor expanded up to second order terms with respect to the stochastical constitutive variables $\tilde{a}$ and $\tilde{b}$ in (4.36) in the following form:

$$\tilde{C}^{(s)}(U^*) = C_0^{(s)}\,(U^*) + \left.\frac{\partial \tilde{C}^{(s)}}{\partial \tilde{a}}\right|_{U^*} \Delta\tilde{a}$$

$$+ \left. \frac{\partial \tilde{C}^{(s)}}{\partial \tilde{b}} \right|_{U^*} \Delta b$$

$$+ \frac{1}{2} \left[\frac{\partial^2 \tilde{C}^{(s)}}{\partial \tilde{U}^2} \Delta \tilde{a}^2 + 2 \frac{\partial^2 \tilde{C}^{(s)}}{\partial \tilde{a} \; \partial \tilde{b}} + \Delta \tilde{a} \; \Delta \tilde{b} \right. \tag{4.38}$$

$$\left. + \frac{\partial^2 \tilde{C}^{(s)}}{\partial \tilde{b}^2} (\Delta \tilde{b})^2 \right]$$

The stochastic element stiffness matrix $\tilde{K}^{(s)}$ was defined as a quadrature of $[B^{(s)}]^T \; [\tilde{C}^{(s)}] \; (B^{(s)}]$ at selected Gauss integration point leading to a form:

$$\tilde{K}^{(s)} = K_0^{(s)} \; (U^*) + \frac{\partial \; \tilde{K}^{(s)}}{\partial \; \tilde{a}^{(s)}} \Delta a^{(s)} + \frac{\partial \tilde{K}^{(s)}}{\partial \tilde{b}^{(s)}} \Delta b^{(s)}$$

$$+ \frac{1}{2} \left[\frac{\partial^2 \tilde{K}^{(s)}}{\partial \tilde{a}^{(s)}} (\Delta a^{(s)})^2 + 2 \frac{\partial^2 \tilde{K}^{(s)}}{\partial \tilde{a}^{(s)} \; \partial \tilde{b}^{(s)}} \Delta a^{(s)} \; \Delta b^{(s)} \right. \tag{4.39}$$

$$\left. + \frac{\partial^2 \tilde{K}^{(s)}}{\partial \; \tilde{b}^{2(s)}} (\Delta b^{(s)})^2 \right]$$

In their derivatives of the stiffness matrix could be computed in . the following form:

$$\frac{\partial \tilde{K}^{(s)}}{\partial \tilde{a}^{(s)}} = \int_v [B^{(s)}]^T \left[\frac{\partial \; \tilde{C}^{(s)}}{\partial \; \tilde{a}^{(s)}} \right] [B] \; dv$$

$$\frac{\partial^2 \; \tilde{K}^{(s)}}{\partial \; \tilde{a}^{(s)} \; \partial \; \tilde{b}^{(s)}} = \int_v [B^{(s)}]^T \left[\frac{\partial^2 \; \tilde{C}^{(s)}}{\partial \; \tilde{a}^{(s)} \; \partial \; \tilde{b}^{(s)}} \right] [B] \; dv \tag{4.40}$$

Thus all partial derivative of the global stiffness matrix $\tilde{K}$ can be obtained by assembling the aforementioned corresponding partial derivatives defined for each element. Along with the Taylor expanded version for $\tilde{K}$-- in the same form (4.39) -- second order perturbation of $\tilde{U}$ in terms of U_0, $\frac{\partial \tilde{U}}{\partial \tilde{a}}$ and $\frac{\partial \tilde{U}}{\partial \tilde{b}}$ leads to the following system of linear equations:

$$K^0 \; (U^{*0}) \; U^) = F \tag{4.41a}$$

$$K^0 \; (U^{*0}) \; \frac{\partial \tilde{U}}{\partial \tilde{a}} + \frac{\partial \tilde{K}}{\partial \tilde{a}} U^0 = 0 \tag{4.41b}$$

$$K^0 \; (U^{*0}) \; \frac{\partial \tilde{U}}{\partial \tilde{b}} + \frac{\partial \tilde{K}}{\partial \tilde{b}} U^0 = 0 \tag{4.41c}$$

$$\frac{\partial^2 \tilde{K}}{\partial \tilde{a} \; \partial \tilde{b}} U^0 + \frac{\partial \tilde{K}}{\partial \tilde{a}} \frac{\partial \tilde{U}}{\partial \tilde{b}} + \frac{\partial \tilde{K}}{\partial \tilde{b}} \frac{\partial \tilde{U}}{\partial \tilde{a}} \tag{4.41d}$$

$$+ K^0 \frac{\partial^2 \tilde{U}}{\partial \tilde{a} \; \partial \tilde{b}} = 0$$

This permits computation of mean and dispersion of $\tilde{U}$ in terms of $\frac{\partial \tilde{U}}{\partial \tilde{b}}$, $\frac{\partial^2 \tilde{U}}{\partial \tilde{a} \; \partial \tilde{b}}$ and in terms of means and dispersions of $\tilde{a}$ and $\tilde{b}$.

Numerical evaluation of the derivatives, according to (4.40) could be somewhat complicated when the nonlinear stress-strain relation (4.37b) is elastoplastic in character as in (4.36). This step will consume substantial computational resources in a large problem.

The paper does not elaborate on numerical implementation of the steps presented therein.

Nakagiri and Hisada applied the perturbation method to evaluate safety and reliability for finite element representation of structural systems. The paper [N-4.5] described a framework to apply the mean and dispersion of response quantities, as evaluated in [N-4.1] through [N-4.4], in order to calculate safety indices. The methodology of the standard reliability technique applied therein can be found in [W-4.1].

The time history analysis with a stochastical description of a proportional damping matrix was presented in [N-6]. Some of the crucial aspects of the latter are described below.

The equation of motion for a damped finite element system can be written to be

$$\tilde{M}\,\ddot{\tilde{U}} + \tilde{K}\,\tilde{U} + \tilde{K}'^{\,*}\,\dot{\tilde{U}} = \tilde{F}(t) \tag{4.42}$$

In the case of proportional damping the damping matrix $\tilde{K}'$ is a linear combination of the mass and stiffness matrices $\tilde{M}$, $\tilde{K}$, in the form:

$$\tilde{K}' = \tilde{a}\,\tilde{M} + \tilde{b}\,\tilde{K} \tag{4.43}$$

In the specific example [N-4.7] the authors considered deterministic mass and stiffness matrices, then $\tilde{M} = M$ and $\tilde{K} = K$ and focused attention on stochasticity of the damping matrix via the random variables $\tilde{a}$ and $\tilde{b}$ in (4.43). In the computational step that was presented in [N-4.7] the authors formulated a broader class of problems where the damping matrix $\tilde{K}'$ was decomposed into

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* conventionally C is used to indicate the damping matrix. In order to avoid confusion with using C for the constitutive matrix in [B-1.4], a nonstandard notation, K' is used to denote the damping matrix herein.

a deterministic component K'_0 and a stochastic part which is proportional damping in nature i.e.:

$$\tilde{K}' = K'_0 + \tilde{a}\, M + \tilde{b}\, K$$

Furthermore, K'_0 was not necessarily in the form of a Cangley series with K and M. Hence K_0 was not reducible to a diagonal matrix with real mode shapes $\{\phi_i\}$ pertaining to those of the undamped system. A generalized version of the aforementioned equation would be

$$\tilde{K}' = K'_0 + \Sigma\, \tilde{a}_i\, K'_i \tag{4.44}$$

where each K'_i would reduce to a diagonal when transformed into the modal coordinates as follows:

$$[\phi]^T [K'_i]\ [\phi] = \langle K''_i \rangle \tag{4.45}$$

where $\langle K''_i \rangle$ is a diagonal matrix. It may be remarked that such a generalization is very appropriate for a wide class of practical problems.

In the formulation a generalized coordinate $\{\tilde{q}_i\}$ was Taylor expanded in terms of the stochastic parameters $\tilde{a}_j$ of (4.44) up to quadratic terms:

$$\{\tilde{q}_i\} = \{q_{0i}\} + \frac{\partial\{\tilde{q}_i\}}{\partial \tilde{a}_j}\, \Delta a_i + \frac{1}{2}\, \Sigma\, \frac{\partial_2\, q_i}{\partial \tilde{a}_j \partial \tilde{a}_\ell}\, (\Delta a_j)\ (\Delta a_\ell) \tag{4.46}$$

substitution of (4.46) and (4.45) in (4.42) led to the following set of equations when the modal representation was sought:

$$m \ddot{q}_i + \phi_i^T k_0' \phi_i q_i + k q_i = \phi_i^T f(t) \tag{4.47a}$$

$$m_i \frac{\partial \ddot{q}_i}{\partial a_j} + \phi_i^T k_0' \phi_i \frac{\partial q_i}{\partial a_j} + k_i \frac{\partial q_i}{\partial a_j} = - k_j'' q_i \tag{4.47b}$$

$$m \frac{\partial^2 \ddot{q}_i}{\partial a_j \partial a_\ell} + \phi_i^T k_0' \phi_i \frac{\partial^2 q_i}{\partial a_j \partial a_\ell} \tag{4.47c}$$

$$+ k \frac{\partial^2 q_i}{\partial q_j \partial a_\ell} = - (k_j'' \frac{\partial q_i}{\partial a_\ell} + k_\ell'' \frac{\partial q_i}{\partial a_j})$$

(note: no sum over repeated indices)

The modal mass and stiffness components were obtained from the mode shapes as:

$$\phi_i^T M \phi_i = M_i \text{ and } \phi_i^T K \phi_i = K_i \tag{4.48}$$

The authors solved the aforementioned set of equations by employing Newmarks implicit time integration scheme ["$\beta = \frac{1}{4}$"]. Numerical results for a tower with fourteen beam elements subjected to El Centro (1940) NS acceleration input was selected to be the input ground motion.

The statistical computation was simplified by assuming the random variables $\tilde{a}$ in (4.44) describing the random coefficient for the proportional damping matrix to be of zero mean Gaussian distribution. Thus the required input were Cov $[a_i, a_j]$. It should be noted that the third moment:

$$E\,[a_i, a_j, a_\ell] = 0 \tag{4.49}$$

and the fourth moment reduced as:

$$E\,[a_i, a_j, a_\ell, a_m] = \text{Cov}\,[a_i, a_j]\ \text{Cov}\,[a_\ell, a_m] + \tag{4.50}$$

$$\text{Cov}\,[a_i, a_\ell]\ \text{Cov}\,[a_j, a_m] + \text{Cov}\,[a_i, a_m]\ \text{Cov}\,[a_j, a_\ell]$$

Numerical results in the form of graphs indicated expectation and ("3-σ") bounds of top deflection and the effects of Cov $[a_i, a_j]$ on the standard deviation of top deflection for the tower problem. These are perhaps the only meaningful numerical results published for dynamic analysis of a finite element system with stochastic damping matrix.

A column buckling problem [N-4.3] with stochastic description of the stiffness matrix $\tilde{K}$ and the geometrical stiffness matrix $\tilde{K}_g$ led to the computation of the buckling load via the following eigenvalue problem:

$$[\tilde{K}]\ \{\tilde{z}\} = \tilde{\lambda}\ [\tilde{K}_g]\ \{\tilde{z}\} \tag{4.51}$$

The buckling load is related to the eigenvalue $\tilde{\lambda}$ and the bent shape is described by the eigenfunction $\{\tilde{z}\}$. It should be remarked that the content of this paper [N-4.4] is identical with [C-3.1] where the free vibration problem:

$$[\tilde{K}]\ \{\tilde{z}\} = \tilde{\omega}^2\ [\tilde{M}]\ \{\tilde{z}\}$$

was described. In (4.52) $\tilde{\omega}$ and $\{\tilde{z}\}$ are the stochastical natural frequency and the mode shape due to stochastic mass and stiffness matrices, $\tilde{M}$ and $\tilde{K}$, respectively.

In the note [N-4.4] the authors presented the problem of buckling of a cantilever beam with the stochastical descriptions of end restraints as shown in the Fig. 4.1:

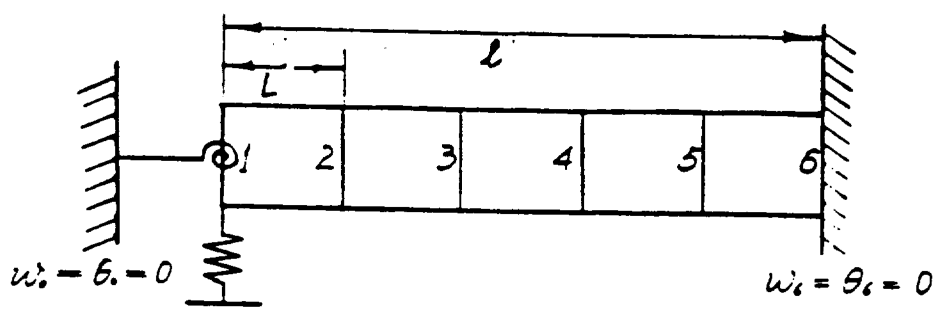

Fig.4.1 Cantilever Beam Buckling Problem

The element stiffness matrix for element number 1 which was attached to a stochastical spring with translational spring constant $\left(\frac{\tilde{s}}{1-\tilde{s}}\right) \frac{EI}{L^3}$ and rotational spring constant $\left(\frac{\tilde{c}}{1-\tilde{c}}\right) \frac{EI}{L}$ would be

$$\tilde{K}^{(1)} = \frac{EI}{L^3} \begin{bmatrix} (12 + \frac{\tilde{s}}{1 - \tilde{s}}) & 6\ell & -12 & 6L \\ & (4 + \frac{\tilde{c}}{1 - \tilde{c}}\ \ell^2 & -6L & 2L^2 \\ \text{symmetric} & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}$$

(4.53)

The geometrical stiffness matrix for each interior element was deterministic in nature and was represented in the usual fashion:

$$K_g^{(s)} = \frac{EI}{30L^3} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ \text{symmetric} & & 36 & -3L \\ & & & 4L^2 \end{bmatrix}$$

(4.54)

In the procedure that followed the element stiffness matrices were assembled in global matrices $[\tilde{K}]$ and $[K_g]$. The stochastic processes, viz $[\tilde{K}]$, $\tilde{\lambda}$, and $\{\tilde{z}\}$ were Taylor expanded up to second order terms with respect to the random parameters $\tilde{s}$ and $\tilde{c}$ about their mean values, s_0 and c_0, respectively. Thus

$$\tilde{K} = K^0 + \frac{\partial \tilde{K}}{\partial \tilde{c}} \Delta c + \frac{\partial \tilde{K}}{\partial \tilde{s}} \Delta s + \frac{1}{2} [\frac{\partial^2 \tilde{K}}{\partial \tilde{c}^2} + (\Delta c)^2\ 2\ \frac{\partial^2 \tilde{K}}{\partial \tilde{s} \partial \tilde{c}} (\Delta s)(\Delta c)$$

$$+ \frac{\partial^2\ \tilde{K}}{\partial\ \tilde{s}^2} (\Delta s)^2] \qquad (4.55)$$

since $\tilde{K}$ in (4.53), had rather simple algebraic expressions in terms of $\tilde{s}$ and $\tilde{c}$ evaluation of the partial derivatives, i.e. α, β tensors of [B-1.4], are indeed straightforward. Thus

$$\frac{\partial \tilde{K}}{\partial \tilde{s}} = \frac{EI}{L^3} \frac{s_0}{(1-s_0} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{Symmetric} & & & 0 \end{bmatrix} \tag{4.56a}$$

$$\frac{\partial^2 \tilde{K}}{\partial \tilde{s}^2} = \frac{EI}{L^3} \frac{s_0^{\,2}}{(1-s_0)^3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \text{Symmetric} & & 0 & 0 \\ & & & 0 \end{bmatrix} \tag{4.56b}$$

$$\frac{\partial \tilde{K}}{\partial \tilde{c}} = \frac{EI}{L^3} \frac{c_0}{(1-c_0)^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & L^2 & 0 & 0 \\ \text{Symmetric} & & 0 & \\ & & & 0 \end{bmatrix} \tag{4.56c}$$

$$\frac{\partial^2 \tilde{K}}{\partial \tilde{c}^2} = \frac{EI}{L^3} \frac{c_0^{\,2}}{(1-c_0)^3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & L^2 & 0 & 0 \\ \text{Symmetric} & & 0 & 0 \\ & & & 0 \end{bmatrix} \tag{4.56d}$$

Now similar expansion, as in (4.55), for the eigenvalue $\tilde{\lambda}$ (which was proportional to the buckling load $\tilde{P}$, $\tilde{P} = \frac{\tilde{\lambda} L^2}{EI}$) and the bent shape $\{\tilde{z}\}$ were carried out in the following form:

$$\tilde{\lambda} = \lambda_0 + \frac{\partial\tilde{\lambda}}{\partial\tilde{s}} \Delta s + \frac{\partial\tilde{\lambda}}{\partial\tilde{c}} \Delta c + \frac{1}{2} \frac{\partial^2\tilde{\lambda}}{\partial\tilde{c}^2} \Delta s^2$$
$$+ \frac{\partial^2 \tilde{\lambda}}{\partial\tilde{c}\partial\tilde{s}} \Delta s \ \Delta c + \frac{1}{2} \frac{\partial^2\tilde{\lambda}}{\partial c^2} \Delta c^2 \tag{4.57}$$

and

$$\tilde{z} = z_0 + \frac{\partial\tilde{z}}{\partial\tilde{c}} (\Delta s)^2 + \frac{\partial^2 \tilde{z}}{\partial\tilde{s}\partial\tilde{c}} (\Delta s) \ (\Delta c) \ \frac{1}{2} \frac{\partial^2\tilde{z}}{\partial\tilde{c}^2} (\Delta c)^2 \tag{4.58}$$

Note eventhough $\frac{\partial^2 \tilde{K}}{\partial\tilde{s}\partial\tilde{c}} = 0$, $\frac{\partial^2 \tilde{\lambda}}{\partial\tilde{s}\partial\tilde{c}}$ and $\frac{\partial^2 z}{\partial\tilde{s}\partial\tilde{c}}$ are nonzero due to coupling through the implicit "inversion" in a linear eigenvalue problem. Substitution of (4.53) through (4.58) in (4.51) led to the following system of linear equations (when the coefficients of Δs, Δc, $(\Delta s)^2$, $(\Delta c)^2$ are set to zero individually)

$$([K_0] - \lambda_0 \ [K_g]) \ \{z_0\} = 0 \tag{4.59a}$$

$$(\frac{\partial\tilde{K}}{\partial\tilde{s}} - \frac{\partial\tilde{\lambda}}{\partial\tilde{s}} K_g) \ \{z_0\} + (K_0 - \lambda_0 \ K_g) \ \{\frac{\partial\tilde{z}}{\partial s}\} = 0 \tag{4.59b}$$

$$(\frac{\partial\tilde{K}}{\partial\tilde{c}} - \frac{\partial\tilde{\lambda}}{\partial\tilde{c}} K_g) \ \{z_0\} + (K_0 - \lambda_0 \ K_g) \ \{\frac{\partial\tilde{z}}{\partial c}\} = 0 \tag{4.59c}$$

$$(\frac{\partial^2\tilde{K}}{\partial\tilde{s}^2} - \frac{\partial^2\tilde{\lambda}}{\partial\tilde{s}^2} K_g) \ \{z_0\} + (\frac{\partial\tilde{K}}{\partial\tilde{s}} - \frac{\partial\tilde{\lambda}}{\partial\tilde{s}} K_g) \ \{\frac{\partial\tilde{z}}{\partial\tilde{s}}\}$$
$$+ (K_0 - \lambda_0 \ K_g) \ \{\frac{\partial^2 z}{\partial\tilde{s}^2}\} = 0 \tag{4.59d}$$

$$\left(\frac{\partial^2\tilde{K}}{\partial\tilde{c}^2} - \frac{\partial^2\tilde{\lambda}}{\partial\tilde{c}^2} K_g\right) \{z_0\} + \left(\frac{\partial\tilde{K}}{\partial\tilde{c}} - \frac{\partial\tilde{\lambda}}{\partial\tilde{c}} K_g\right) \{\frac{\partial\tilde{z}}{\partial\tilde{c}}\} + (K_0 - \lambda_0 K_g) \{\frac{\partial^2\tilde{z}}{\partial\tilde{c}^2}\} = 0 \tag{4.59e}$$

$$- \frac{\partial^2 \tilde{\lambda}}{\partial\tilde{s}\partial\tilde{c}} K_g \{z_0\} + \left(\frac{\partial\tilde{K}}{\partial\tilde{s}} - \frac{\partial\tilde{\lambda}}{\partial\tilde{s}} K_g\right) \frac{\partial\tilde{z}}{\partial\tilde{c}} + \left(\frac{\partial\tilde{K}}{\partial\tilde{c}} - \frac{\partial\tilde{\lambda}}{\partial\tilde{c}} K_g\right) \frac{\partial\tilde{z}}{\partial\tilde{s}} + (K_0 - \lambda_0 K_g) \frac{\partial^2 \tilde{z}}{\partial\tilde{s}\partial\tilde{c}} = 0 \tag{4.59f}$$

It is to be noted that computation of partial derivatives such as $\frac{\partial\tilde{\lambda}}{\partial\tilde{s}}$, $\frac{\partial^2 \tilde{z}}{\partial\tilde{s}\partial\tilde{c}}$ etc. in the aforementioned equation, can be carried out by solving the generalized version of the linear eigenvalue problems of the form:

$$\lambda \{U\} + [A] \{z\} + \{V\} = 0 \tag{4.60}$$

where λ (scalar) and $\{z\}$ (vector) are unknowns but $\{U\}$ and $\{V\}$ (vectors) and $[A]$ (matrix) are prescribed.

The authors presented numerical results for a sample case and demonstrated the accuracy of this second order perturbation method.

Finally, the mean the dispersion of $\tilde{\lambda}$ and $\{\tilde{z}\}$ were obtained following the methods in the previous paper [N-4.1], [N-4.2] and [N-4.3].

Nakagiri and Hisada also employed the perturbation technique to the specific cases of stochastic Winkler foundation [N-4.6] and for random misfit in frame structures [N-4.8]. These two papers are of peripheral importance in describing the general procedure of the stochastic finite element formulation with perturbation techniques.

In a conference paper [N-4.9] the authors summarized the perturbation technique developed in [N-4.1] to [N-4.8]. Numerical results for two specific problems were presented. The expected value and dispersion of stress intensity factor for an edge crack with uncertain length were presented in graphical form. The second problem dealt with mean and standard deviation of inplane stress developed in a long strip. The authors compared the results of the first and second order approximations (where the stochastic processes were Taylor expanded up to linear and quadratic terms, respectively). In certain cases the difference of result was quite significant. Hence the authors recommended the formulation with second order approximations.

Handa [H-4.1] initiated stochastical finite element calculations to enhance design analysis of civil engineering structures. In a sequence of research reports and conference papers [H-4.2], [H-4.3], [H-4.4] Handa and his associates employed finite element analysis technique to estimate expected values and correlation coefficients of static stresses and displacements for trusses, frames and beam structures. Stochastical variations of structural section geometry, material property as well as that of applied loading were considered. All their discussions were restricted lognormal distribution of stochastical parameters. For example, for a finite element (s) the carrying capacity $R^{(s)}$ and any load effect $S^{(s)}$ were assumed to be lognormally distributed. This facilitated the construction of the safety margin z defined to be:

$$z^{(s)} = \ln R^{(s)} - \ln S^{(s)} \tag{4.61}$$

to have a normal distribution.

The presentation [H-4.1] detailed out the first order perturbation method. The authors remarked that the error associated with neglecting the higher order terms will not exceed 20% at most. In the interest of brevity the steps are not repeated here since more detail algebraic development are presented in this section in equations (4.1) through (4.16).

The authors presented several numerical examples of the first order second moment formulation. Two noteworthy cases among those will be summarized here.

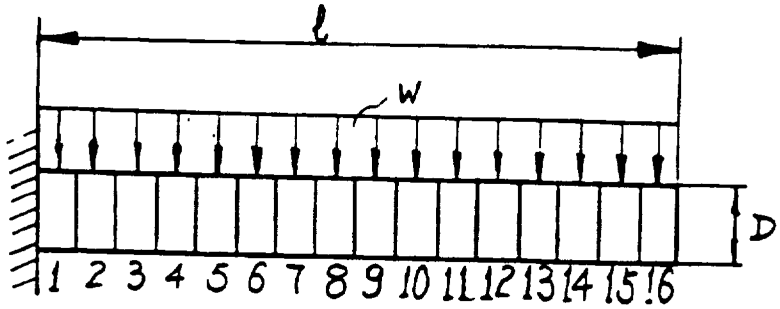

Fig.4.2 Cantilever Beam Problem

A cantilever steel beam, as shown in Fig. 4.2, with deterministic length L and diameter D, was analyzed with the following uncorrelated stochastical processes:

(i) loading W(x),

(ii) second moment of area I(x) and

(iii) modulus of elasticity E(x).

The mean values W_0, I_0, E_0 and the standard deviations σ_W, σ_I and σ_E were prescribed as input data. The spatial variability of the aforementioned stochastical processes was taken to be exponential. The autocorrelation functions for such processes were expressed in the form:

$$\rho\ (x_i, x_j) = \exp\ (-\kappa\ |x_i - x_j|) \tag{4.62}$$

The constant κ (with dimension of length inverse) was anticipated to be obtained experimentally. Numerical computations were carried out by assigning $\kappa = 0$ (fully correlated), $\kappa \to \infty$

(uncorrelated) and intermediate $\kappa = 2$ (partially correlated) cases. It was demonstrated that the standard deviation for the displacement of the tube at the free end was significantly dependent upon the spatial variability criteria as depicted by the correlation coefficient in Fig. 4.3.

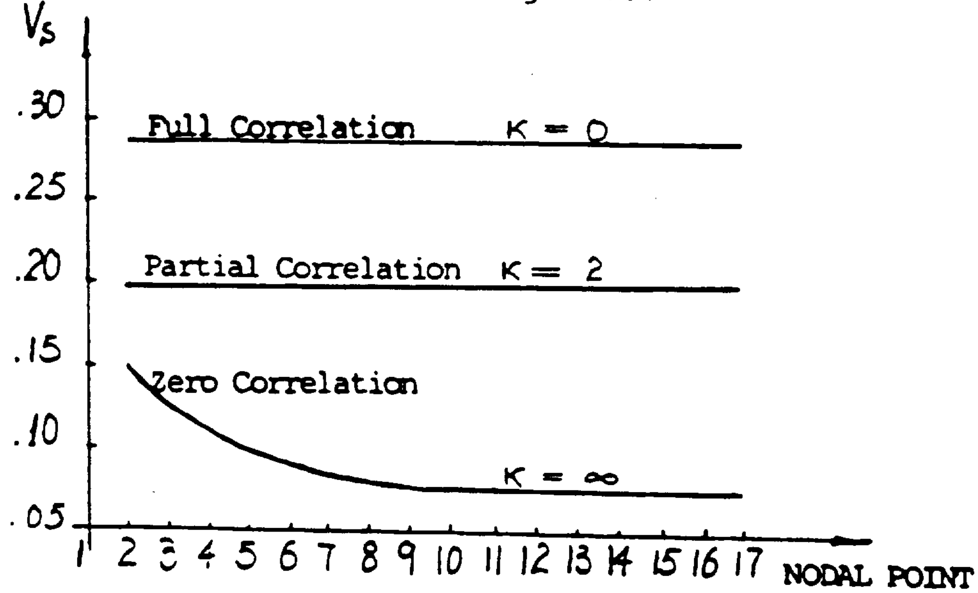

Fig.4.3 Effect of Correlation

The second numerical example in [H-4.4] dealt with a two-dimensional framed truss as shown in Fig. 4.4.

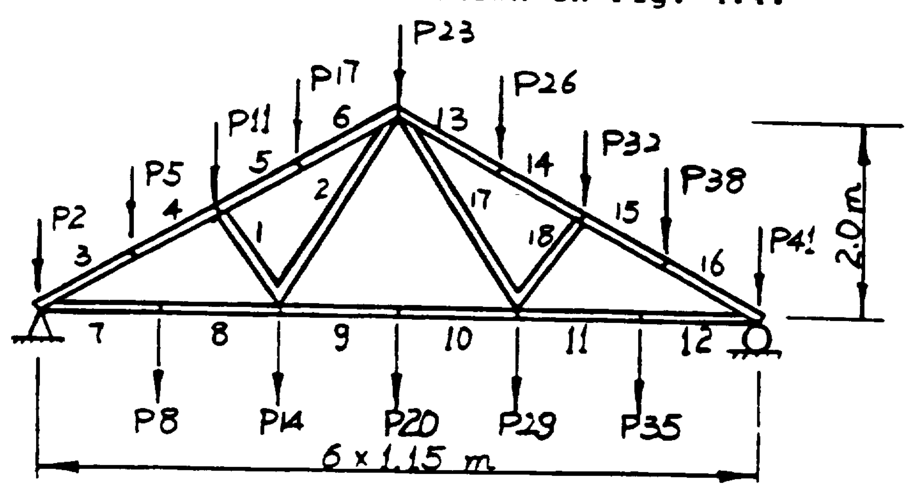

Fig.4.4 Framed Truss Problem

There were 21 degrees-of-freedom. The area of cross-sections, second moment of areas and the moduli of elasticity were considered as random variables (with prescribed means and standard deviations). The stochasticity of fully correlated loading was expressed by a fully populated 14 x 14 matrix. Correlation between cross-sectional area, second moment of inertia and moduli of elasticity i.e. Cov[A,E], Cov[A,I], Cov[I,E] were taken to be zero. The variance matrix for each random quantity (associated with those 18 nodes) was thus obtained in the form of a diagonally dominated banded matrix.

The authors used a computer program [H-4.3] to carry out the stochastic finite element calculations. Mean values and covariance coefficients of displacements of all nodes and stresses in each element were evaluated. The correlation matrix for stresses was fully populated but decayed with geometrical distance between two elements.

Scale of Fluctuation Method

Vanmarcke conducted extensive research [V-4.1], [V-4.2], [V-4.3], [V-4.4], on engineering problems dealing with partial differential equation of mathematical physics where the properties of the domain are random processes in the spatial coordinates. The outstanding contribution was to systemically develop expressions for local spatial averages of stochastical quantities as well as variances. For example, in one-dimensional situation, a stochastic process X_i is a "random function" of the spatial coordinate x: $\tilde{X}_i(x)$. In general the expected value for the product $\tilde{X}_i(x_1)$ and $\tilde{X}_i(x_2)$ will be

$$E\ [\tilde{X}_i(x_1)\ \tilde{X}_i(x_2)] = \mathrm{Cov}\ [\tilde{X}_i,\ x_1, x_2] \quad \text{or} \quad = \mathrm{Cov}\ [\tilde{X}_i,\ |x_2 - x_1|,\ x_1] \tag{4.63}$$

For <u>stationary processes</u> (with respect to spatial coordinates) the values in (4.63) are independent of x_1 and could be expressed as

$$E\ [\tilde{X}_i\ (x_1)\ \tilde{X}_i\ (x_2)] = \mathrm{Cov}\ [\tilde{X}_i,\ |x_2 - x_1|] \tag{4.64}$$

or

$$E\ [\tilde{X}_i\ (x_1),\ \tilde{X}_i\ (x_2)] = \sigma_{\tilde{X}_i}^2\ \rho_{\tilde{X}_i}\ (x_2 - x_1) \tag{4.65}$$

where ρ is the autocorrelation function.

Within the framework of finite element analysis the distributions over an element "i" are "smeared out" and statistical average quantities were defined over the element domain L_j in the form

$$\tilde{X}_{ij\ \text{jth element}} = \frac{1}{Lj} \int_{Lj} \tilde{X}_i\ (\underset{\sim}{x})\ dv \tag{4.66}$$

Now $\tilde{X}_{ij}$ is the <u>equivalent random variable</u> over jth element associated with the <u>stochastical process</u> $\tilde{X}_i(x)$. If $\tilde{X}_i$ is a wide-sense stationary process then one defines the mean $m_{\tilde{X}_{ij}}$ and variance Var $[\tilde{X}_{ij}]$ in the form:

$$E\ [\tilde{X}_{ij}] = m_{\tilde{X}_i}$$

$$\text{Var}\ [\tilde{X}_{ij}] = \sigma^2_{\tilde{X}_i}\ \gamma_{\tilde{X}_i}\ (L_j) \tag{4.67}$$

in which $\gamma_{\tilde{X}_i}\ (L_j)$ described the dependence of the element size on variance function. From the identities of random signal processing one could write:

$$\gamma_{\tilde{X}_i}\ (x) = \int_0^x \int_0^x \rho_{\tilde{X}_i}\ (x_1 - x_2)\ dx_1\ dx_2 \tag{4.68}$$

Furthermore, the variance function $\gamma_{\tilde{X}_i}(x)$ has the property:

$$\gamma_{\tilde{X}_i}\ (0) = 1 \text{ and } \underset{x \to \infty}{\alpha t}\ \gamma_{\tilde{X}_i}(x) = 0 \tag{4.69}$$

The principal contribution of Vanmarcke's presentation is to define the <u>scale of fluctuation</u> $\theta_{\tilde{X}_i}$ based on the asymptotic behavior of γ in the following form:

$$\gamma_{\tilde{X}_i}(x) = \frac{\theta_{\tilde{X}i}}{x} \text{ as } x >> \theta_{\tilde{X}_i} \tag{4.70}$$

and

$$\theta_{\tilde{X}_i} = 2 \int_0^\infty \rho_{\tilde{X}_i}(x)\ dx \tag{4.71}$$

whenever the above limit (4.70) and the integral (4.71) exist. An important interpretation of $\theta_{\tilde{X}_i}$ is in terms of the Wiener-Kinchine spectral density function $g_{\tilde{X}_i}(\omega)$ where

$$g_{\tilde{X}_i}(w) = \frac{2}{\pi} \int_0^{\infty} \rho_{\tilde{X}_i}(x) \cos \omega x \, dx \tag{4.72}$$

and then

$$\theta_{\tilde{x}_i} = \pi g_{\tilde{x}_i}(0) \tag{4.73}$$

It is important to note that during the selection of mesh size, for a stochastic process indicated by Wiener-Kinchine spectral density the characteristic length of a finite element region should be less than the scale of fluctuation.

Considerable development regarding the scale of fluctuation for general two-, three- and n- dimensional random processes could be found in the textbook [V-1.1]. Vanmarcke also presented very useful approximate formulae for the scale of fluctuations based on the asymptotic behavior of the correlation function. Detail mathematical development for unidirectional and two-dimensional random variates along with useful algebraic identities could be found in the conference paper [V-4.1].

The journal paper [V-4.2] described in detail the problem of static deformation of an idealized shear beam shown in Fig. 4.5.

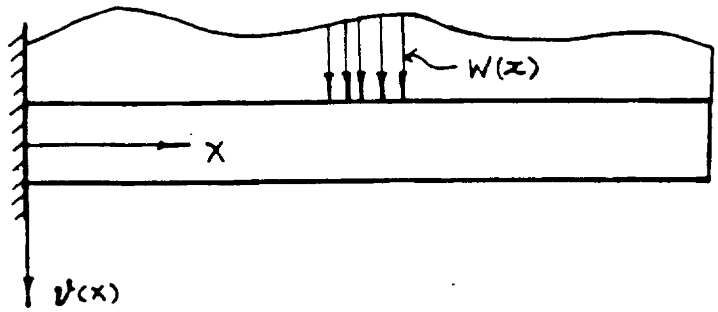

Fig.4.5 Shear Beam Problem

The continuum equation for the deflection v(x) was expressed as

$$\frac{dv(x)}{dx} = \alpha(x)\ V(x) \tag{4.74}$$

in which V(x) was the shear force and $\alpha(x)$ was the shear flexibility. (For a rectangular beam of area A(x) and shear modulus G(x) the shear rigidity is given by G(x)A(x).)

Then

$$\alpha(x) = 1/\ G(x)\ A(x)) \tag{4.75}$$

For the fixed end beam since v(x=0) = 0

$$v(x) = \int_0^x \frac{dv}{dx}\,dx = \int_0^x \alpha(x)\ T(x')\ dx' \tag{4.76}$$

Vanmarcke and Grigoriu illustrated the problem with deterministic $T(x')$. The shear flexibility factor $\tilde{\alpha}(x)$ was assumed to be a stationary random process in the spatial variable x with a mean m_α and variance $\sigma_\alpha{}^2$, autocorrelation function $\rho_2(x)$ and scale of fluctuation Θ_α. The integral relation relating the random displacement $\tilde{v}(x)$ was given by

$$\tilde{v}(x) = \int_0^x \tilde{\alpha}(x')\, T(x')\, dx' \tag{4.77}$$

The above equation indicates a linear transformation of T(x) into V(x). Thus

$$E[\tilde{v}(x)] = E[\int_0^x \tilde{\alpha}(x')\, T(x')\, dx'] \tag{4.78a}$$

$$= m_\alpha \int_0^x T(x')\, dx' \tag{4.78b}$$

and

$$\mathrm{Var}\,[\tilde{v}(x)] = \sigma_\alpha^2 \int_0^x \int_0^x \rho_\alpha\,(x_1' - x_2')\, T\,(x_1')\, T\,(x_2^1)\, dx_1'\, dx_2' \tag{4.79}$$

Some special cases of interest would be

(i) completely correlated case $\rho_{\tilde{\alpha}}(x) = 1$ (4.80)

$$\mathrm{Var}\,[\tilde{v}(x)] = \sigma_{\tilde{\alpha}}{}^2\, [\int_0^x T\,(x')\, dx']^2 \tag{4.81}$$

(ii) uncorrelated case

$$\rho_{\tilde{\alpha}}(x) = \delta(x) \quad [\delta(x): \text{ Dirac's delta} \tag{4.82}$$

then

$$\text{Var } [\tilde{v}(x)] = \sigma_{\tilde{\alpha}}^2 \int_0^x [T(x')]^2 \, dx' \tag{4.83}$$

The aforementioned evaluations of the Var $[\tilde{v}(x)]$ for the general case (4.79) was simplified by introducting the notion of scale of fluctutation $\Theta_{\tilde{\alpha}}$.

Vanmarcke and Grigoriu considered the one-dimensional shear beam case (refer to Fig. 4.5) for the stochastic variable of shear flexibility $\tilde{\alpha}(x)$ subjected to two loading cases. For the concentrated load P, applied at the end of the cantilever beam the expected value and the variances were obtained in the following closed forms:

$$E\ (\tilde{v}_N) = m_{\tilde{\alpha}} \, PL \tag{4.84}$$

$$\text{Var } (\tilde{v}_N) = \sigma_{\tilde{\alpha}}^2 \, P^2 \, L^2 \, \gamma_{\tilde{\alpha}} (L) \tag{4.85}$$

In the case of a uniformly distributed (deterministic) load p_0 the authors furnished:

$$E\ [\tilde{v}(x)] = m_{\tilde{\alpha}} \, p_0 \, L \, (x - \frac{x^2}{2L}) \tag{4.86}$$

$$\mathrm{Var}\ [\tilde{v}(x)] = \sigma_{\tilde{\alpha}}^2\ p_0^2\ L^2 \int_0^x dx_1^1 \int_0^x (L-x_1^1)(L-x_2^1).$$

$$\rho_{\tilde{\alpha}}\ (x_1^1 - x_2^1)\ d\ x_2^1 \tag{4.87}$$

Numerical results in graphical forms summarize the variation of the standard deviation of the end displacement with $L/\theta_{\tilde{\alpha}}$, for various autocorrelation functions such as

$$\rho_{\tilde{\alpha}}(x_1,x_2) = \exp\ (-\pi\ |\frac{x_1-x_2}{\theta}|^2) \tag{4.88a}$$

$$\rho_{\tilde{\alpha}}(x_1,x_2) = \exp\ (-2\ |\frac{x_1-x_2}{\theta}|^2) \tag{4.88b}$$

$$\rho_{\tilde{\alpha}}(x_1,x_2) = 1 + 4\ |\frac{x_1-x_2}{\theta}| + \exp\ (-4\ |\frac{x_1-x_2}{\theta}|) \tag{4.88c}$$

The crucial steps in a stochastic finite element modeling would then be:

Step - 1: Divide a domain D into elements D_i and define the element flexibility quantities as averages:

$$\alpha_i = \theta \int_{D_i} \tilde{\alpha}(x)\ dv \ / \int_{D_i} dv \tag{4.89}$$

Step - 2: Calculate the mean and covariant matrices:

$$\{m_{\alpha_1},\ m_{\alpha_2}\ \ldots m_{\alpha_n}\} = m_{\tilde{\alpha}}\ \{1,1,1,\ \ldots\} \tag{4.90}$$

and Cov $(\alpha_i,\alpha_j) = \Sigma$ (4.91)

For one-dimensional cases the authors used the approximate expression:

$$\text{Cov }(\alpha_i,\alpha_j) = \frac{\sigma_{\tilde{\alpha}}^2}{2} \{(k-1)^2\ \gamma_a\ [\frac{(k-1)\ L}{N}] \tag{4.92}$$

$$- 2k\ \gamma_a\ [\frac{kL}{N}] + (k+1)^2\ \gamma_a\ [\frac{(k+1)\ L}{N}]\} \tag{4.93}$$

Where $k = |i-j|$ and the beam of length L was divided into N equal segments.

Step - 3: The nodal loads Ω_i are to be defined by introducing the shape functions:

$$\Omega_i = \int_{D_i} N_i\ T(x)\ dv \tag{4.94}$$

$$\Omega = \{\Omega_i\} \tag{4.95}$$

Then for the specific case of the shear beam

$$\begin{aligned} v_i = \{&\Omega_1,\Omega_2,\ \ldots\ \Omega_i,\ 0,0\}^T \\ \{&\alpha_1,\alpha_2,\ \ldots\qquad\quad \alpha_N\} \end{aligned} \tag{4.96}$$

Then the expected end displacement and Variance of the nodal displacement vector:

$$E\ [v_i] = m_{\tilde{\alpha}}\ \{\Omega_1,\ \Omega_2\ \ldots\ \Omega_N\} \tag{4.97}$$

$$\mathrm{Cov}\ [v_i, v_j] = \Omega^T\ \Sigma\ \Omega \tag{4.98}$$

The approximation for the covariance matrix Σ will depend on $\sigma_{\tilde{\alpha}}$ and the scale of fluctuation $\theta_{\tilde{\alpha}}$ for a selected number of discretization N.

This paper [V-4.2] will serve as a basis to approximate the covariance matrix on the basis of scale of fluctuation. In practical finite element mesh design the scale of fluctuations for the random field will guide the selection of mesh size.

5. Finite Element Stochasticity by Simulation

Astill, Nosseir and Shinozuka [A-5.1] completed the problem of wave propagation through a random medium by employing a finite element modeling. Instead of resorting to the pertubation method to develop the stochastic mass and stiffness matrices the authors utilized a direct Monte Carlo simulation procedure. This could be the first published paper where displacement, strain and stress histories were generated by solving the dynamic euation of motion of a finite element model with stochastic parameters. The authors focused their attention on the impact problem and captured the effcts of randomness in the contitutive properties on the propagation of stress pulses.

The authors modeled a concrete cylinder with 64 axisymmetric rings. A quadrant was modeled with 85 nodes. Uniform stress impact in the form of a triangular shaped pulse was considered. Graphs of propagation of the stress pulses were presented at various sections.

Spatial variability of Young's modulus E and density ρ were considered. Numerically 100 test samples were recreated by employing the Monte Carlo simulation technique. Sample realizations of E and ρ were plotted against the corresponding mean values. Deviations from the deterministic case for the axial stress distribution were also displayed. The means and standard deviations for the octahedral shearing stress and miximum shearing stress were calculated using the simulated population. Since these physical quantitites govern failure conditions in concrete cylinders the example is indeed of practical interest.

The method of simulation for two one-dimensional random processes $f_1(z)$ and $f_2(z)$ (say the density and the ultimate strength which could vary only axially along z-axis) was presented in satisfactory detail. For homogeneous processes the cross-correlation matrix was defined as:

$$\begin{bmatrix} E\,[f_1(z)\ f_1(z+\zeta)], & E\,[f_1(z)\ f_2(z+\zeta)] \\ \text{symmetric}, & E\,[f_2(z)\ f_2(z+\zeta)] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1{}^2 r_{11}(\zeta) & \sigma_1\sigma_2 r_{12}(\zeta) \\ \text{symmetric} & \sigma_2{}^2 r_{22}(\zeta) \end{bmatrix} \qquad (5.1)$$

In there $r_{ij}(z)$ are the normalized auto-correlation functions. This matrix can be estimated from experimental data. The Wiener-Khinchine transform of the above correlation matrix is the following cross-spectral density:

$$\begin{bmatrix} \sigma_1^2 g_{11}(\eta) & \sigma_1\sigma_2 g_{22}(\eta) \\ \text{Hermetian} & \sigma_2{}^2 g_{22}(\eta) \end{bmatrix} \qquad (5.2)$$

in which

$$g_{ij}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r_{ij}(\zeta) \exp(-i\eta\zeta)\, d\zeta$$

Implentation of FFT (Fast Fourier Transform) based algorithm led to the simulated variables for homogeneous multivariate Gaussian processes. The density and crushing strengths were obtained by simulation and then the Young's modulus for each sample was calculated by their nonlinear transformation (as is common in concrete failure analysis).

Computer code to carry out conventional finite element dynamic calculations was proposed whereas very sophisticated simulation techniques were used to generate sample finite element system (mass, stiffness) matrices. In order to adhere to the prescribed spatial distribution of stochastic processes, which represent randomness of material properties, the authors constructed cross-spectral density matrices. The required mathematical treatment demands thorough training in computational statistics. It should be remarked that merely ad hoc generation of realization for system matrices will prove to be completely useless. In structural reliability assessment the randomness of the system should be viewed in the light of multi-variate and multi-dimensional processes [S-5.1]. Gaussian processes with ARMA (AutoRegressive Moving Average) representation [S-5.2] are very useful indeed. However, for non-Gaussian stochasticity the computational complexity and the requirement of theoretical background in computational statistics could make an analysis almost prohibitive at the existing level of technology.

The aforementioned paper by Astill, Nosseir and Shinozuka could be the only complete treatment on simulation to be of practical significance. Engineers undertaking Monte Carlo simulation for spatially varying random processes will find that presentation extremely useful. It should be remarked that the proposed simulation technique demands advanced training in computational statistics especially in random process analysis. However, the method to generate statistics (means, dispersions, etc.) is straightforward once simulation techniques are mastered. Thus the appropriate steps will be:

(i) to obtain a realization of geometrical and material properties, etc., according to design statistical criteria;

(ii) to carry out conventional finite element analysis;

(iii) to generate a population by repeating (i) and (ii)

(iv) to construct mean, covariance matrix, skewness, etc. from the results of (iii).

The mathematical treatment of Monte Carlo simulation is arousing new interest since the emergence of parallel processors, [K-7.1]. Research is underway to reformulate simulated finite element models in order to take the advantage of inherent parallelism in finite element formulation [S-8.1].

6. Papers of Special Interest

Der Kiureghian applied the finite element method to analyze reliability aspects of linear structures consisting of random variables [D-6.1 and D-6.2]. Consistant with the notion of computing the performance index of a structure, a stochastical vector $\tilde{S}$ is defined to represent the effects of random load and random system properties. Any response quantity, like stress, deformation, can be included in this vector. The paper elaborates the first-order reliability approach, which relates the $\tilde{S}$ vector with the allowable "strength" variables vector R (which typically include design stresses, tolerable deformations, etc.). Description of the stochastic finite element formulation, as applied to a linear static system, is extremely clear in the presentation. It may be remarked the [D-6.1] and [D-6.2] are perhaps the only two papers in the field of probabilistic finite elements, where all the conclusions and statements are substantiated with numeric developments. The papers are devoid of conjectures and ad hoc promises regarding the computability of large systems with probabilistic variables. The beam example presented in these two papers [D-6.1] and [D-6.2] which are essentially the same, is summarized below.

A beam element in a two-dimensional configuration was described with three degrees-of-freedom at each end. Associated with these translational and rotational deformations, the static stiffness matrix for the uniform section was described by the following stiffness matrix.

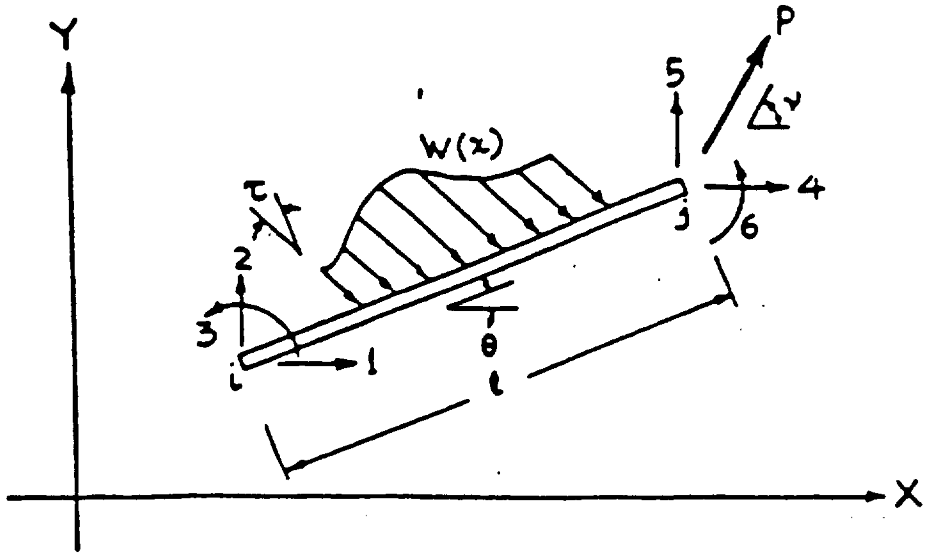

Fig.6.1 Definition of Two-Dimensional Beam Element

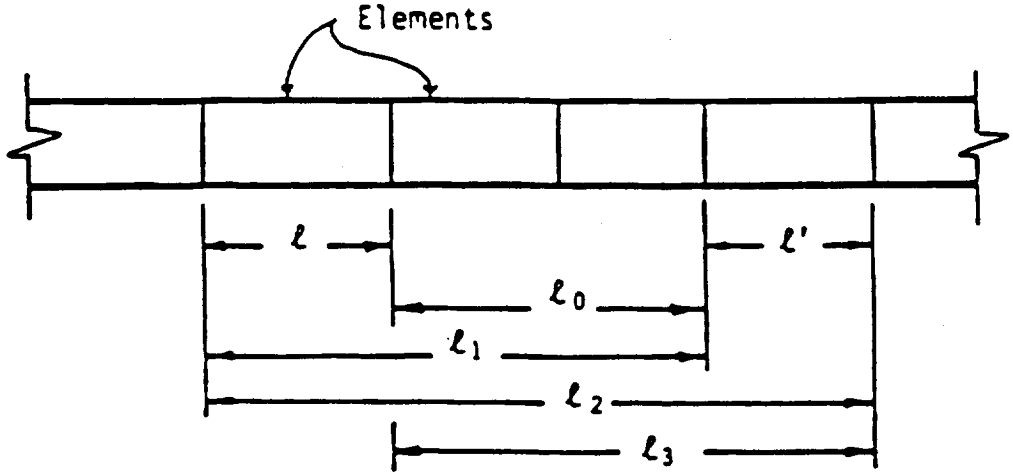

Fig.6.2 Finite Element Discretization

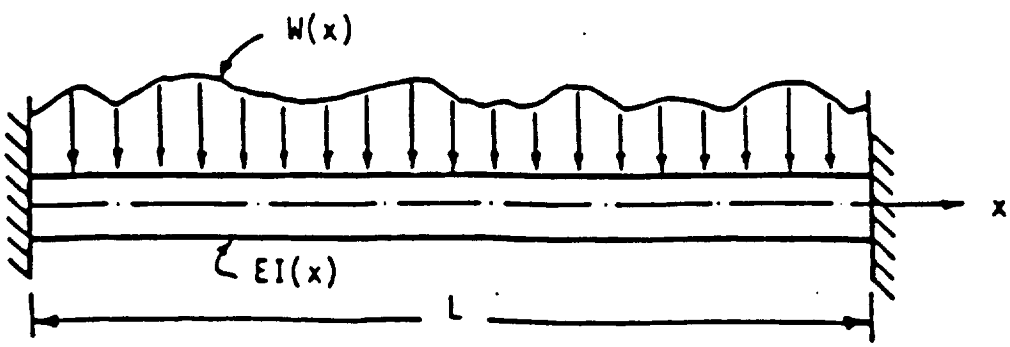

Fig.6.3 Fixed-fixed Beam Problem

$$
[K^{(s)}] = \begin{bmatrix} a & & & & & \\ b & d & & & & \\ c & e & f & & \text{symmetric} & \\ -a & -b & -c & a & & \\ -b & -d & -e & b & d & \\ c & e & g & -c & -e & f \end{bmatrix} \tag{6.1}
$$

where

$$
\begin{aligned}
a &= \frac{12EI}{L^3}\sin^2\theta + \frac{EA}{L}\cos^2\theta, \quad b = (\frac{EA}{L} - 12\,\frac{EI}{L^3}\sin\theta\cos\theta \\
c &= \frac{6EI}{L^2}\sin\theta, \quad d = \frac{12EI}{L^3}\cos^2\theta + \frac{EA}{L}\sin^2\theta \\
e &= \frac{6EI}{L^2}\cos\theta, \quad f = \frac{4EI}{L}, \quad g = \frac{2EI}{LL}
\end{aligned} \tag{6.2}
$$

and E = modulus of elasticity, A = area, I = moment of inertia. The authors also described the required partial derivatives such as the $\alpha_{ij\ell}$ tensors, when variability of the material property E and cross-section A or second-moment of the area I are to be accounted for. The "form" of the matrices remain the same. In the case of the $\frac{\partial K^{(s)}}{\partial E}$ calculation, the ij-th element can be directly obtained as $\frac{1}{E}\,K^{(s)}{}_{ij}$. Simplified expression for $\frac{\partial K^{(s)}}{\partial A}$ can be written with $a = \frac{E}{L}\cos^2\theta$, $b = \frac{E}{L}\sin\theta\cos\theta$, $d = \frac{E}{L}\sin^2\theta$ and $c = e = f = g = 0$, in (6.1). Similar calculations are possible for $\frac{\partial K^{(s)}}{\partial I}$ with $a = \frac{12E}{L^3}\sin^2\theta$, $b = -\frac{12E}{L^3}\sin\theta\cos\theta$, $c = -\frac{6E}{L^2}\sin\theta$, $d = \frac{12E}{L^3}\cos^2\theta$, $e = \frac{6E}{L^2}\cos\theta$, $f = \frac{4E}{L}$, $g = \frac{2E}{L}$, in (6.1)

The aforementioned formulation <u>does not permit</u> the possibility of <u>random description of the nodal point</u>

coordinates. In order to allow the end locations (x_i, y_i) and (x_j, y_j) to assume a spatial variation character, one needs to use in (6.2)

$$\theta = \tan^{-1} \frac{y_j - y_i}{x_j - x_i} \qquad (6.3a)$$

and

$$\ell = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \qquad (6.36)$$

This makes the $\frac{\partial K^{(s)}}{\partial x_i}$ type formulas much more cumbersome than those which appear in (6.2).

The authors describe the force vector $F^{(e)}$ due to a uniformly distributed load W, which could be calculated by using the finite element shape functions. This entailed the calculation of $\frac{\partial F^{(e)}}{\partial W}$ quantity as:

$$\frac{\partial F^{(e)}}{\partial W} = \begin{matrix} a\cos\phi + b\sin\phi \\ a\sin\phi - b\cos\phi - c\cos\phi \\ a\cos\phi + b\sin\phi \\ a\sin\phi - b\cos\phi - c\cos\phi \end{matrix} \qquad (6.4)$$

where $a = \frac{L\sin\theta}{2}$, $b = \frac{L\cos\theta}{2}$, $c = \frac{L^2}{12}$, and ϕ is the inclination of load W as depicted in Fig. 6.1.

An important step in the papers is to describe the mean and dispersion of the nodal load due to spatial variability of the load distribution W(x). One utilizes the definition of the nodal load:

$$F_i = \int_0^L W(x)N_i(x)dx \tag{6.5}$$

where $N_i(x)$ is the i-th shape function along x. The expected value operator E was applied to the above equation leading to

$$E[F_i] = \int_0^L E[W(x)]N_i(x)dx \tag{6.6}$$

consequently,

$$E[F_i \cdot F_j] = \int_0^L \int_0^L E[W(x_1)W(x_2)]N_i(x_1)N_j(x_2)dx_1dx_2 \tag{6.7}$$

Thus the mean and autocorrelation function for nodal loads were defined.

The authors focused their attention on Gaussian homogeneous processes. If the loading function W(x) is a Gaussian process, then each element F_i is normally distributed. For non-Gaussian processes, distributions for F_i will pose computational difficulties. From a reliability point of view, one may have to restrict the computation correct up to the second-moment terms.

The authors have developed a completely documented computer code FORAFS to carry out stochastical finite element analysis of frame structures. With the prescription of correlation coefficients of the basic variables, such as member area, moment of inertia, etc., exceedance coefficients can be calculated according to the first-order reliability method by using that computer program.

Liu, Belytschko and Mani [L-6.1] considered a general finite element representation of a dynamic system in the form:

$$M\ddot{\tilde{U}} + \tilde{f}(\tilde{X},\tilde{U},\dot{\tilde{U}}) = F(t) \tag{6.8}$$

where the deterministic mass matrix M is considered with the deterministic load vector F(t). The elastic restoring force $\tilde{f}$ is also dependent on the state vectors: $\tilde{U}$ = displacement and $\dot{\tilde{U}}$ = velocity. The paper displays very encouraging numerical results when compared with Monte Carlo simulation data.

The paper makes the drastic simplification of ignoring the off-diagonal terms in the covariant matrix $\mathrm{Cov}[X_i,X_j]$. For any finite element system with spatial variability considerations, the nonzero correlation distance (which depends upon how fast the correlation coefficients die out, i.e., on the bandwidth of the correlation matrix) is indeed a key statistical consideration. Diagonalization of the covariant matrix indeed simplifies the algebra but is unrealistic for any nontrivial stochastical process with spatial variability.

The algebraic derivation presented therein can be obtained directly from the Nakagiri and Hisada papers [N-4.1] and [N-4.3] when $\mathrm{Cov}[X_i,X_j]$ is assumed in the form $\delta_{ij}\ \mathrm{Var}[X_i]$, where δ_{ij} is Kronecker's delta.

The authors do not detail the simulation technique for nonlinear systems. As has been pointed out in various papers (for example, refer to [P-1.1]. Such numerical simulation is not exactly a routine procedure. Secondly, the assumption of Gaussian distribution for spring constants as in the paper [L-6.1] (when the possibility of negative stiffeners is acceptable as a realization) is quite questionable. The paper, at best, could serve as an example to test out a code under the aforementioned stringent restrictions of covariant matrix to be a diagonal one.

In this category of papers of special interest the most original contribution is by Contrearas, [C-6.1]. The state space representation of dynamic response for a stochastic finite element system was conceived to be a finite dimensional Markov process. The mathematical treatment is elegant and practical even though rather involved. Algebraic details of the paper will be summarized in the review of advanced methods. The following key steps are provided to establish a resemblance of time marching scheme in finite element temporal solution to a Markov process (where the present state depends only on the previous one <u>not</u> on the <u>entire</u> past history).

The author proposed the state vector $\tilde{Y}$ to house the random variables $\tilde{X}$,besides the usual case of displacement $\tilde{U}$ and velocity $\dot{\tilde{U}}$. Thus

$$\tilde{Y} = \begin{Bmatrix} \tilde{X} \\ \tilde{U} \\ \dot{\tilde{U}} \end{Bmatrix} \tag{6.9}$$

The equation of motion of a stochastical dynamic system was written

$$\dot{\tilde{Y}} = f[\tilde{Y}, t] + G[\tilde{Y}, t]\ \tilde{W}r \qquad (6.10)$$

in which f is the discrete operator describing the dterministic equation of motion, G is to represent the contribution of a unit white noie and W is a white noise. The stochastic vector differential equation (6.10) corresponds to Ito's form [A-1.1]. The following temporally discrete form was then obtained:

$$\tilde{Y}\ (t_{n+1}) = \Phi\ \tilde{Y}(tn),\ t_{n+1},\ tn\}$$

$$+ \Gamma\{\ \tilde{Y}(tn),\ tn\}\ W(t_{n+1})$$

where the discrete operators Φ and Γ were obtained in terms of the finite element mass, damping and stiffness matrices. Finally, the unknown vector $\tilde{Y}$ was estimated according to Kalman filtering method.

7. Outstanding Issues and Recommended Research

The major computational concern for a successful execution of a finite element code with system stochasticity could be assessed separately for the two separate techniques, viz. the simulation and the perturbation (and related) methods.

(i) Monte Carle simulation demands the execution of a deterministic conventional finite element code many many times. Each input realization (like material properties for each element, boundary node coordinates, temperature distribution, etc.) is required to be generated by a statistical package independent of the finite element program. A number of paper mainly authored by Shinozuka and his associates, [S-7.1], address the question of simulation in design-analysis for structural engineering problems. The theoretical background for spatially uniform multi-dimensional and multivariate Gaussian processes is quite complete. There are some computer programs available for research purposes which are suitable for finite element models with limited number of degrees-of-freedom. There is indeed a need to develop robust versions of these simulation programs.

In the case of nonGaussian and nonstationary processes research work is urgently required for the successful development of simulation algorithms. There is hardly any documented statistical package on the market in order to generate Monte Carlo database for arbitrary distributions (which could be prescibed in tabular observations) particularly suitable for discrete structural analysis.

There is no available dynamic finite element code which is integrated with a simulation procedure. Engineers, who are trained in computational mechanics and simultaneously possess working knowledge in applied statistics should be entrusted to develop such a statistical front end to a finite element computer program.

Computers with pipeline architecture and those aided with parallel processors can carry out repeated deterministic calculations of simulation in a faster, more accurate and much more economical fashion, refer to [K-7.1]. The Computational Statistics group of the Society for Industrial and Applied Mathematics (SIAM) organizes meetings to present the state-of-the-art procedures on simulation. Attention is drawn to those highly theoretical and analytically oriented formulations for Monte Carlo technique. Useful and practical computational tools for discrete realizations (as demanded in a finite element input data stream) could be developed on the basis of the aforementioned modern mathematical research.

(ii) A direct finite element modeling on a stochastic input database invariably necessitates a Taylor series expansion at certain stages of computation. Thus the perturbation principle is quite inherent in such formulaions. In various presentations computation of the deviator such as ΔK [$= \tilde{K}-K_0$], refer to [B-1.4], or the derivatives with respect to random variables $\tilde{X}$ such as $\partial\tilde{K}/\partial\tilde{X}_i$ become essential. This will demand rewriting of stiffness routines in a conventional finite element computer code. These routines are much more lengthy, especially when higher derivatives such as $\beta_{ij\ell m} = \partial^2\tilde{K}_{ij}/\partial\tilde{X}_\ell \partial X_m$ are required.

For example, in simple truss problems with stochastic nodal point coordinates, the expression for the derivative of the element stiffness matrix $K^{(s)}$ with respect to nodal coordinates x_i, y_i, etc., involves lengthy expressions in terms of trigonometric inverse functions and their derivatives. The chore to hand calculate such derivatives and then to develop FORTRAN subroutines will be extremely time consuming for complicated finite elements, such as shell elements.

The reviewer himself uses a computer algebra program SMP (Symbolic Manipulation Program), [S-7.2], to formulate finite element stiffness matrices and to evaluate derivatives with respect to the algebraic variables in the stiffness routines. The Taylor expansion routine in SMP is quite handy. It is possible to obtain readily the algebraic expressions of the stochastic stiffness matrix $\tilde{K}$ expanded about the mean K_0 when the closed form expression for $K^{(s)}$ is prescribed. It is very strongly recommended that attention is paid to integrate finite element FORTRAN program with algebraic manipulating softwares in order to develop versatile computer code for Probabilistic Finite Element Method (PFEM). It is to be recognized that initial experience with the computational philosophy of SMP would demand a substantial research effort. In the long run, the code development activities could be expedited for PFEM and PSAM programs when those powerful algebraic software tools are implemented.

In selecting the computer algebra program the reviewer recommends SMP over another similar package called MACSYMA. From mathematical point of view SMP is much more versatile and is particularly efficient for tensor analysis. Stiffness matrices, and especially their derivatives constitute higher rank tensors which are amenable to SMP programming. Systematic and user friendly manipulations of those algebraic entities are more suited to SMP than to MACSYMA.

The specific requirement for PFEM computer program is the "post-processing calculation" to estimate statistics (like mean, standard deviation, skewness, etc.) of response quantities (such as displacements, stresses, etc.) obtained according to a finite element formulation. The method essentially converts a continuum field problem into sets of matrix equations such as the strain ($\tilde{\varepsilon}$) - displacement ($\tilde{U}$) relation:

$$\tilde{\varepsilon} = [\tilde{B}]\ \tilde{U} \qquad (7.1)$$

the force ($\tilde{F}$) - displacement - ($\tilde{U}$) equation:

$$\tilde{F} = [\tilde{K}]\ \tilde{U} \qquad (7.2)$$

Hence the generic problem is

$$\tilde{Y} = [\tilde{A}]\ \tilde{X} \qquad (7.3)$$

where the statistics of $\tilde{X}$ are either prescribed or computed from the finite element calculations. The matrix $[\tilde{A}]$ is in general nonlinear in $\tilde{X}$. The question is then how to estimate statistics of $\tilde{Y}$ for a large correlated $[\tilde{A}]$. The complexity of such expressions like (7.3) can be readily witnessed even in a linear elastostatic problem where the entries in the element stiffness matrix is a highly nonlinear function in stochastic Poisson's ratio. It is a significant computational task to obtain the numerical values of correlation coefficients for the elements of stiffnes matrices, when the probability distribution function of the Poisson's ratio is prescribed, is a significant numerical task. Attention should be focused on developing a computer program which could yield such statistics under general nonGaussian prescriptions of fundamental stochastical variables.

The final answer sought from a PFEM computer program is the prediction for exceedence. Reliability based methods developed by Wirshing and his associates (W-7.1] adequately address that point. Statisticians have developed Pearson series and related exponential families to incorporate skewness and high order moments in exceedence calculations.

The adequacy of second moment based exceedence predictions should be examined by carefully designed Monte Carlo simulation. Higher order moments could contribute significantly in nonlinear problems especially in the case of large displacements and large strain situations. Parametric studies are recommended for benchmark problems in order to gain confidence in multivariate nonGaussian and nonstationary processes. Time history analysis for stochastic systems with a large number of degrees-of-freedom is essential to test the finite element codes generated in PSAM, PFEM programs.

8. Non-structural Applications

Problems of both system schostacity and random forcing function appeared in different problems of engineering science. In electrical and electronic communications, the noise elimination aspects in signal processing deal with random processes in time. In fluid mechanics random turbulence is of interest. These problems are quite closely related to those of structural mechanics.

Biostatisticians in stochastic biometrics have developed algorithms very closely related to the stochastical finite elements discussed herein. Bookstein [B-8.1] used both triangular and quadrilateral elements in a mesh to study statistical effects of growth in space time continuum. Goodall [G-8.1] applied statistical methods in a highly coupled system of nonlinear partial differential equations to predict plant growth. Their contributions are particularly important since they discuss bases of computational methods to solve the discrete analog with random system parameters. (Their work will be summarized in the literature review of Advanced Method.) Shinozuka and Moss-Salenteijn [S-8.1] applied the Monte Carlo simulation technique to predict growth of long bones in mammals.

An interesting application of stochastic analysis on a discrete system (not really a finite element per se) deals with the description of branding in trees, [A-8.1], the Markov process which describes the stochastic behavior of the growth continuum degenerates into Fibonacci number series in the discrete model.

9. Conferences Related to PSAM

International Conferences on Applications of Statistics and Probability to Soil and Structural Engineering (Computational Methods dealing with probabilistic structural analysis). The proceedings are available in bound volumes from the host institutes. The first meeting [P-9.1] was in Hong Kong, September 13 to 16, 1971 and the fourth (last) meeting [P-9.2] took place in Florence, Italy, June 13-17, 1983.

The ICOSSAR (International Conference on Structural Safety and Reliability) addressed the question of Structural analysis according to probabilistic considerations and accomodated analysis of discrete systems with stochastic variables. Some important contributions which were presented at the third conference, [N-4.1] and [H-4.4] on stochastic finite elements are reviewed here in sections 4. In the near future the fourth conference at Kobe, Japan, will have a session on Stochastic Finite Elements. The notable authors are Vanmarcke, Mochio, Shinozuka, Hisada, Nakagiri, and Der Kiureghian. Some of their research papers are reviewed in sections 4, 5 and 6.

The National Science Foundation sponsored a recent conference 1984, on [C-2] Water Resources where the finite element method was applied to nondeterministic systems. From the viewpoint of non-structural applications useful. These papers are useful.

The ASCE-EMD committee on Probabilistic Methods is arranging a session on Stochastical Finite Element Methods at the joint

ASCE and ASME summer conference at Albuquerque, New Mexico, for June 23-26, 1985. The notable speakers are: Der Kiureghian, Wen, Lawrence, Ang, Shinozuka, Grigoriu, Khater and O'Rourke. Since 1969, ASCE also has sponsored a series of Specialty Conferences on stochastic mechanics and structural reliability, i.e. at Purdue University 1969, Stanford University 1974, Tuscon, Arizona, 1979 and in 1984 at U.C. Berkeley. Proceedings are available as ASCE publication. The majority of the papers are on stochastic loading rather than on system stochasticity.

One major aspect of organizing a successful PFEM code is to include the state-of-the-art research in computational statistics. It should be remarked that engineers, who are so competent in computational mechanics, hardly demonstrate significant contribution or appreciation for the research in probabilistic developments. SIAM organized two conferences with short courses in computational statistics, the first was in Boulder, Colorado, June 11, 1984 and the second in Boston, Massachusetts, October, 21, 1984. Many of the discrete probabilistic formulations in PFEM have their close analog in related branches. Computational tools such as interactive graphics, use of parallel processing, employment of artificial intelligence in heuristic solution, would play a significant role in those analytical formulations. The future review on Advanced Methods will address the related topics such as: Nonlinear Optimization in Statistical Variational Formulations, Multi-objective Optimization Algorithms in Discrete Computations, Graphical Methods in Computational Statistics, Monte Carlo simulations in Supercomputers and Statistical Issues and Uncertainty in Artificial Intelligence for discrete stochastic systems. Conferences organized by the American Statistical Association (ASA), International Association for Scientific Computing (IASC) and SIAM will be reviewed.

10. References

*[A-1.1]: Arnold, L. Stochastic Differential Equations : Theory and Applications, Wiley, New York, NY, 1974.

[A-5.1]: Astill, C.J., Nosseir, S.B. and Shinozuka, M., "Impact Loading on Structures with Random Properties," Journal of Structural Mechanics, Vol. 1, No. 1, pp63-77, 1972.

[A-8.1]: Agu, M. and Yokoi, Y., "A Stochastic Description of Branching Structures of Tress," Journal of Theoretical Biology, Vol. 112, pp 667-676, 1985.

[B-1.1]: Benjamin, J.R. and Cornell, C.A., Probability, Statistics and Decision for Civil Engineers, McGraw-Hill, New York, NY, 1970

[B-1.2]: Bolotin, V.V., Statistical Methods in Structural Mechanics, Holden-Day, San Francisco, California, 1969

[B-1.3]: Bracewell, R., The Fourier Transform and its Application, McGraw-Hill, New Yorw, NY, 1965.

[B-1.4]: Burnside, O.H., Probabilistic Finite Element Analysis Applied to Static Linear Elastic Systems, Southwest Research Institute, PSAM report to NASA, December, 1984.

[B-1.5]: Beaumont, G.P., Introductory Applied Probability, Halsted Press, John Wiley, NY, 1983.

[B-3.1]: Boyce, W.E., "Buckling of a Column with Random Initial Displacement," Journal of Aero Science, Vol 18, pp308-320, 1961.

[B-8.1]: Bookstein, F.L., "A Statistical Method for Biological Shape Comparisons," Journal of theoretical Biology, Vol. 107, pp 475-520, 1984.

[C-1.1]: Chernov, L.A., Wave Propagation in a Random Medium, Dover, New York, NY, 1967

[C-1.2]: Clarkson, B.L. (editor), Stochastic Problems in Dynamics, Pitmar Press, London, U.K., 1977.

[C-1.3]: Cramer, H. and Leadbetter, M.R., Stationary and Related Stochastic Processes, Wiley, New York, NY, 1967.

*—Reference Format [X-m.n]: X-Author, m-Section, n-Sequence.

[C-1.4]: Crandell, S.H., Mark, W.D., Random Vibration in Mechanical Systems, Academic Press, New York, NY, 1963.

[C-3.1]: Collins, J.D. and Thomson, W.T., "The Eigenvalue Problem for Structural Systems with Stochastical Properties," AIAA Journal, Vol. 7, No. 4, pp 642-648, 1969.

[C-3.2]: Cambou, B., "Application of First Order Uncertainity Analysis in the Finite Element Method in Linear Elasticity," Proceedings, Second International Conference - Application of Statistics and Probability in Soil and Structural Engineering," London, England, 1973.

[C-6.1]: Contreras, H., "the Stochastic Finite Element Method," International Journal Computers and Structures, Vol. 12, pp 341-348,1980.

[D-1.1]: Drake, A.W., Fundamentals of Applied Probability Theory, McGraw-Hill, New York, NY, 1967.

[D-1.2]: Daverpart, W.B., and Rort, W.L., Introduntion to the Theory of Random Signals and Noise, McGraw-Hill, New York, NY, 1958

[D-6.1]: Der Kiureghian, A., "Finite Element Methods in Structural Safety Studies," Proceedings, Symposium on Structural Safety Studies, ASCE Convention, Denver, Colorado, April, 1985.

[D-6.2]: Der Kiureghian, A. and Ke, J.B., "Finite Element Based Reliability Analysis of Frame Structures," Proceedings, International Conference on Structural Safety and Reliability, Kobe, Japan, May, 1985.

[E-1.1]: Elishakoff, I., "Hoff's Problem in a Probabilistic Setting," Journal Applied Mechanics, ASME, Vol. 47, No. 2, pp 403-408, June, 1980. [F-1.1]: Frendenthal, A.M., "The Safety of Structures," ASCE Transactions, Vol. 112, pp 125-180, 1947.

[G-1.1]: Goel, N.S. and Richterdyn, N., Stochastic Models in Biology, Academic Press, New York, NY, 1974.

[G-1.2]: Gumbe3l, E.J., Statistics of Extremes, Columbia University Press, New York, NY, 1958.

[G-8.1]: Gooodall, C., "The Statistical Analysis of Two-dimensional Growth," Doctoral Dissertation, Harvard University, 1983.

[H-1.1]: Heyman, D.P. and Sobel, M.J., Stochastic Models in Operations Research, Volumes I and II, McGraw-hill, NY, 1983 and 1984.

[H-4.1]: Handa, K., "Application of FEM in the Statistical Analysis of Structures," Division of Structural Design, Chalmers Unriversity of Technology, Gotegorg 1975, reprot, 1975:6.

[H-4.2]: Handa, K. and Karrhold, G., "Application of Lognormal Distribution Function in the Statistical Analysis of Structures," Division of Structural Design, Chalmers University of Technology, Gotegorg 1978, report 1978:9.

[H-4.3]: Handa, K. and Andersson, K., SLUMPFEM, Division of Structural Design, Chalmers University of Technology, goteborg 1978, report 1978:26.

[H-4.4]: Handa, K. and Andersson, K., "Application of finite Element Methods in Stochastical Analysis of Structures," Proceedings, ICOSSAR '81, 3rd International Conference - Structural Safety and Reliability, Trondheim, Norway, June 23-25, 1981.

[K-4.1]: Khinchin, A.Y., "Korrelationstheorie der Stationaren Stochastischer, Mathematics Analysis, Vol. 109, pp605-615, 1934.

[K-7.1]: Kalos, M.H., "Monte Carlo Methods and Computers of the Future," Proceedings, ASA-IASC-SIAM Conference - Frontiers in Computational Statistics, Bostion, October, 1984.

[L-1.1]: Lin, Y.K., Probabilistic Theory of Structural Dynamics, McGraw-Hill, New York, NY, 1967.

[L-1.2]: Lin, Y.K., Random Processes, McGraw-Hill, New York, NY, 1969.

[L-1.3]: Jumley, J.L., Stochastic Tools in Turbulence, Academic Press, New York, NY, 1970.

[L-6.1]: Liu, W.K., Belytschko, T. and Mani, A., "A Computational Method for the Determination of the Probabilistic Distribution of the Dynamic Response of Structures," Proceeding, ASME Summer conference, New Orleans, June, 1985.

[M-3.1]: Mak, C.K.K. and Kelsey, S., "Statistical Aspects in the Analysis of Structures with Random Imperfections," Proceedings, First International Conference - Applications of Statistics and Probability to Soil and Sructural Systems, Hong Kong, September 13-16, 1971.

[N-4.1]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-1) - Variation of Stress and Strain Caused by Shape Fluctuations," SEISAN-KENKYU, Vol. 32, No. 2, pp 39-42, 1980.

[N-4.2]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-2) - Variation of Stress and Strain Caused by Fluctuations of Material Properties and Geometrical Boundary Conditions," SEISAN-KENKYU, Vol. 32, No. 5, pp 28-31, 1981.

[N-4.3]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-3) - An Extension of the Methodology to Nonlinear Problems," SEISAN-KENKYU, Vol. 32, No. 12, pp 14-17, 1980.

[N-4.4]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-4) - "Eigenvalue Problem of Column Buckling under Uncertain Boundary Conditions," SEISAN-KENKYU, Vol. 33, No. 7, pp 28-31, 1981.

[N-4.5]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-5) - "A Framework for Structural Safety and Reliability," SEISAN-KENKYU, Vol. 33, No. 7, pp 14-17, 1982.

[N-4.6]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-6) - "An Application in Problems of Uncertain Elastic Foundation," SEISAN-KENKYU, Vol. 35, No. 1, pp 20-23, 1983.

[N-4.7]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-7) - "Time-history Analysis of Structural Vibration," SEISAN-KENKYU, Vol. 35, No. 5, pp 26-29, 1983.

[N-4.8]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-8) - "An Application to Uncertain Intrinsic Stresses Generated in Frame Structure with Misfits," SEISAN-KENKYU, Vol. 35, No. 7, pp 32-35,1983.

[N-4.9]: Nakagiri, S., and Hisada, T., "A Note on Stochastic Finite element Method - (Part-8) - "Stochastic Finite Element Method Developed for Structural Safety and Reliability," Proceedings, 3rd ICOSSAR '81, pp 395-408, 1981.

[P-1.1]: Parkus, H., (editor), Random Excitation of Structures by Earthquake and Atmospheric Turbulence, CISM Courses and Lectures, No. 225, Springer Verlag, New York, NY, 1977.

[P-1.2]: Parzen, E., Stochastic Process, Holden-Day, San Francisco, California, 1962.

[P-9.1]: Proceedings, First International Conference - Applications of Statistics and Probability to Soil and Sructural Systems, Hong Kong, September 13-16, 1971.

[P-9.2]: Proceedings, Second International Conference - Applications of Statistics and Probability to Soil and Sructural Systems, London, England, 1972.

[P-9.3]: Proceedings, Third International Conference - Applications of Statistics and Probability to Soil and Sructural Systems, Sydney, Australia, January 19 - February 2, Published by Unisearch, NY, 1979.

[P-9.4]: Proceedings, Fourth International Conference - Applications of Statistics and Probability to Soil and Sructural Systems, Flerence, Italy, June 13-17, 1983.

[P-9.5]: Proceedings, ICOSSAR '81, International Conference - Structural Safety and Reliability, Trondheim, Norway, June 23-25, 1981.

[P-9.6]: Proceedings, ICOSSAR '85, International Conference - Structural Safety and Reliability, Kobe, Japa, May 27-29, 1985.

[P-9.7]: "Probabilistic Mechanics and Structural Reliability," ASCE Engineering Mechanics and Structure Division, University of Arizona, Tuscon, January 10-12, 1979.

[S-1.1]: Schweppe, F.C., Uncertain Dynamic Syste4ms, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[S-5.1]: Shinozuka,M. "Time and Space Domain Analysis in the Structural Safety and Reliability," Proceedings, Second International Conference - Structural Safety and Reliability, Technische Universitat, Munich, W. Germany, pp. 9 - 28, 1977.

[S-5.2]: Samaras, E., Shinozuka, M. and Tsurui, A., "ARMA Representation of Random Processes," Journal of Engineering Mechanics, Vol. 111, No. 3, pp 449-461, 1985.

[S-7.1]: Shinozuka, M., "Keynote Lecture, Freudentthal Memorial Session," Proceedings, ICOSSAR '85, Kobe, Japan, May, 1985.

[S-7.2]: SMP, A Symbolic Manipulation Program, Reference Manual, Inference Corporation, Las Angeles, California, 1983.

[S-8.1]: Skalak, R., "Analysis of Growth and Form," Program Project Proposal to the National Institute of Health, NIH Grant 1-P01-HD-19446.

[S-9.1]: Shinozuka, M., Tsurui, A., Naganuma, T., Moss, M. and Moss-Salentijn, L., "A Stochastic-Mechanical Model of Longitudinal Long Bone Growth," Journal of Theoretical Biology, Vol. 108, pp 413-436, 1984.

[T-1.1]: Tribus, M., Rational Description, Decisions and Desings, Pergamon Press, New YOrk, NY, 1969.

[V-1.1]: Vanmarcke, E.H., Random Fields: Analysis and Synt M.I.T. Press, Cambridge, Mass., 1983.

[V-4.1]: Vanmarcke, E.H., "Probabilistic Modeling for Soil Profiles," Journal of the Geotechnical Ehgineerin Division, ASCE, Vol. 103, Nov. 1977, pp 1227-12

[V-4.2]: Vanmarcke, E.H., "Reliability of Earth Slopes," J of the Geotechnical Engineering Division, ASCE, V 103, Nov. 1977, pp1247-12265.

[V-4.3]: Vanmarcke, E.H., "On the Scale of Fluctuation of F Functions," Research Report R79-19, M.I.T., Civil Engineering Department, Cambridge, Mass, 1979.

[W-4.1]: Wirsching, P.H. "Mechanical Reliability and Probabilistic Design," Literature Review, NASA/Lew Project on Probabilistic Structural Analysis Metho (PSAM), February, 1985.

[W-7.1]: Wirsching, P.H., and Wu, Y.-T., "A Review of Moder Approaches to Fatigue Reliability Analysis and Desi Random Fatigue Life Prediction, ASME Fourth Nationa Congress on Pressure Vessel and Piping Technology, Portland, OR, June, 1983.

[W-8.1]: Whittle, P., "On Stationary Processes in the Plane," Biometrica, Vol. 41, 1954, pp 434-499.

[W-9.1]: Wiener, N., "Generalized Harmmanic Analysis," Acta Mathematics, Vol. 55, 1930, pp117-258.

[Y-1.1]: Yaglom, M., An Introduction to the Theory of Station Random Functions, Dover, New York, NY, 1973.

[Z-1.1]: Zienkiewicz, O.C., The Finite Element Method, Third Edition, McGraw-Hill, New York, NY, 1977.

Section 3

Level 2 PFEM Formulation Applied to Static Linear Elastic Systems

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LEVEL 2 PFEM FORMULATION APPLIED TO STATIC LINEAR ELASTIC SYSTEMS

1.0 Index Notation for Problem Formulation

The following discussion casts, using index notation, the Level 2 finite element formulation for the stochastic displacement vector, element strains and stresses in terms of the random load vector and the random variables in the stiffness matrix.

From Reference [1] the stochastic displacement vector may be written as

$$\begin{aligned} \tilde{U} = U_o &+ I\,[K_o^{-1}\Delta F - K_o^{-1}\Delta K U_o] \\ &- (K_o^{-1}\Delta K)^1\,[K_o^{-1}\Delta F - K_o^{-1}\Delta K U_o] \\ &+ (K_o^{-1}\Delta K)^2\,[K_o^{-1}\Delta F - K_o^{-1}\Delta K U_o] \\ &- (K_o^{-1}\Delta K)^3\,[K_o^{-1}\Delta F - K_o^{-1}\Delta K U_o] \\ &+ \\ &\vdots \end{aligned} \tag{1}$$

where U_o = deterministic displacement vector

K_o^{-1} = inverse of deterministic stiffness matrix

ΔK = random stiffness matrix measured from the deterministic state

ΔF = random force vector measured from the deterministic state

The matrix K_o^{-1} is denoted as

$$K_{oij}^{-1} = f_{oij} \tag{2}$$

Reference [1]: Appendix to SwRI Monthly Technical Letter to NASA-LeRC, date of publication: January 30, 1985.

which is the deterministic flexibility matrix. The size of this square matrix is ixj.

The matrix ΔK_{ij} will be evaluated by expanding the stochastic stiffness matrix $\tilde{K}_{ij}$ in a Taylor series of powers of the random variable vector $\Delta \underline{X}$ about the deterministic solution. For terms up to third order in $\Delta \underline{X}$

$$\Delta K_{ij} = \sum_{\ell=1}^{p} \left. \frac{\partial \tilde{K}_{ij}}{\partial \tilde{X}_\ell} \right|_{\underline{X}_o} \Delta X_\ell$$

$$+ \frac{1}{2} \sum_{\ell=1}^{p} \sum_{m=1}^{p} \left. \frac{\partial^2 \tilde{K}_{ij}}{\partial \tilde{X}_\ell \partial \tilde{X}_m} \right|_{\underline{X}_o} \Delta X_\ell \Delta X_m \tag{3}$$

$$+ \frac{1}{6} \sum_{\ell=1}^{p} \sum_{m=1}^{p} \sum_{n=1}^{p} \left. \frac{\partial^3 \tilde{K}_{ij}}{\partial \tilde{X}_\ell \partial \tilde{X}_m \partial \tilde{X}_n} \right|_{\underline{X}_o} \Delta X_\ell \Delta X_m \Delta X_n + \ldots$$

p is the number of random variables in the stiffness matrix. This expression for ΔK_{ij} may be written in simplified index notation as

$$\Delta K_{ij} = \alpha_{ij\ell} \Delta X_\ell + \beta_{ij\ell m} \Delta X_\ell \Delta X_m$$
$$+ \gamma_{ij\ell mn} \Delta X_\ell \Delta X_m \Delta X_n \tag{4}$$

where

$$\alpha_{ij\ell} = \left. \frac{\partial \tilde{K}_{ij}}{\partial \tilde{X}_\ell} \right|_{\underline{X}_o} \tag{5a}$$

$$\beta_{ij\ell m} = \frac{1}{2} \left. \frac{\partial^2 \tilde{K}_{ij}}{\partial \tilde{X}_\ell \partial \tilde{X}_m} \right|_{\underline{X}_o} \tag{5b}$$

$$\gamma_{ij\ell mn} = \frac{1}{6} \left. \frac{\partial^3 \tilde{K}_{ij}}{\partial \tilde{X}_\ell \partial \tilde{X}_m \partial \tilde{X}_n} \right|_{\underline{X}_o} \tag{5c}$$

Note that values of the terms $\alpha_{ij\ell}$, $\beta_{ij\ell m}$ and $\gamma_{ij\ell mn}$ are fixed for a specific problem since they are the partial derivatives of the stochastic stiffness matrix $\tilde{K}_{ij}$ evaluated at the deterministic state $\underline{X}_o$.

In equation (1), one recurring term is the matrix product $K_o^{-1}\Delta K$. An element ij of this matrix is denoted as

$$K_o^{-1}\Delta K(ij) = K_{oiq}^{-1}\Delta K_{qj} \tag{6}$$

which may be written using equation (4) as

$$K_o^{-1}\Delta K(ij) = K_{oiq}^{-1}\left(\alpha_{qj\ell}\Delta X_\ell + \beta_{qj\ell m}\Delta X_\ell \Delta X_m + \gamma_{qj\ell mn}\Delta X_\ell \Delta X_m \Delta X_n\right) \tag{7}$$

Higher power of $K_o^{-1}\Delta K$ may be formed using equation (7). For example

$$(K_o^{-1}\Delta K)^2 = \left[K_o^{-1}\Delta K(ij)\right]\left[K_o^{-1}\Delta K(jk)\right] \tag{8}$$

where appropriate changes must be made in the subscripts in equation (7).

We will now turn our attention to the term $\left[K_o^{-1}\Delta F - K_o^{-1}\Delta K U_o\right]$ in equation (1). The i-th component of the vector $K_o^{-1}\Delta F$ be written using index notation as

$$K_o^{-1}\Delta F(i) = K_{oij}^{-1}\Delta F_j \tag{9}$$

Likewise, the i-th component of the vector $K_o^{-1}\Delta KU_o$ becomes

$$K_o^{-1}\Delta KU_o(i) = K_{oiq}^{-1}\Delta K_{qj}U_{oj}^{*}$$

$$= K_{oiq}^{-1}\big(\alpha_{qj\ell}\Delta X_\ell + \beta_{qj\ell m}\Delta X_\ell \Delta X_m \tag{10}$$

$$+ \gamma_{qj\ell mn}\Delta X_\ell \Delta X_m \Delta X_n\big)U_{oj}$$

The expression $\left[K_o^{-1}\Delta F - K_o^{-1}\Delta KU_o\right]$ was denoted in equation (17) [Ref 1] as the vector ΔU_1. Its i-th component may be written using equations (9) and (10) as

$$\Delta U_1(i) \equiv \left[K_o^{-1}\Delta F - K_o^{-1}\Delta KU_o\right]$$

$$= K_{oij}^{-1}\Delta F_j - K_{oiq}^{-1}\big(\alpha_{qj\ell}\Delta X_\ell + \beta_{qj\ell m}\Delta X_\ell \Delta X_m \tag{11}$$

$$+ \gamma_{qj\ell mn}\Delta X_\ell \Delta X_m \Delta X_n\big)U_{oj}$$

We will call equation (11), the <u>first order term in K_o^{-1}</u> because it only involves linear terms in this matrix. Likewise, from equation (1), the second order term in K_o^{-1} for the vector component $\Delta U_2(i)$ may be written as

$$\Delta U_2(i) = -\left(K_o^{-1}\Delta K\right)^1\left[K_o^{-1}\Delta F - K_o^{-1}\Delta KU_o\right]$$

$$= -K_o^{-1}\Delta K(ij)\Delta U_1(j) \tag{12}$$

$$= -K_{oiq}^{-1}\big(\alpha_{qj\ell}\Delta X_\ell + \beta_{qj\ell m}\Delta X_\ell \Delta X_m$$

$$+\gamma_{qj\ell mn}\Delta X_\ell \Delta X_m \Delta X_n\big)\Delta U_1(j)$$

*Note: The subscript o always represents the deterministic state. It does not represent an index of summation.

$\Delta U_1(j)$ is evaluated from equation (11) with the appropriate subscript change of j for i.

The total stochastic displacement vector $\tilde{U}$ is the sum of the terms

$$\tilde{U} = U_o + \Delta U_1 + \Delta U_2 + \Delta U_3 + \ldots \tag{13}$$

where the following recursive relationship exists

$$\begin{aligned} \text{where} \quad \Delta U_1 &= I\left(K_o^{-1}\Delta F - K_o^{-1}\Delta K U_o\right) \\ \Delta U_2 &= -\left(K_o^{-1}\Delta K\right)\Delta U_1 \\ \Delta U_3 &= -\left(K_o^{-1}\Delta K\right)\Delta U_2 \\ &\vdots \\ \Delta U_i &= -\left(K_o^{-1}\Delta K\right)\Delta U_{i-1} \end{aligned} \tag{14}$$

The next part of the discussion presents an approach as to how the stochastic strains and stresses may be computed from the stochastic displacement vector. In general the element strains are computed from the global displacement vector and the strain-displacement matrix. If $\tilde{\varepsilon}^{(s)}$ represents the strain vector in element s, then

$$\tilde{\varepsilon}^{(s)} = \tilde{B}^{(s)}\tilde{U} \tag{15}$$

where $\tilde{B}^{(s)}$ is the element strain-displacement matrix. $\tilde{B}^{(s)}$ is stochastic if randomness can enter in the structural geometry. We will take

$$\tilde{B}^{(s)} = B_o^{(s)} + \Delta B^{(s)} \tag{16}$$

where $B_o^{(s)}$ is the deterministic strain-displacement matrix. It seems consistent with the method of computing ΔK [see equation (3)] to evaluate $\Delta B^{(s)}$ by expanding $\tilde{B}^{(s)}$ in a Taylor series about the deterministic state.

This gives

$$\Delta B_{ij}^{(s)} = \sum_{\ell=1}^{p} \frac{\partial \tilde{B}_{ij}^{(s)}}{\partial \tilde{X}_\ell}\Bigg|_{\underline{X}_o} \Delta X_\ell + \frac{1}{2} \sum_{\ell=1}^{p} \sum_{m=1}^{p} \frac{\partial^2 \tilde{B}_{ij}^{(s)}}{\partial \tilde{X}_\ell \partial \tilde{X}_m}\Bigg|_{\underline{X}_o} \Delta X_\ell \Delta X_m$$

$$+ \frac{1}{6} \sum_{\ell=1}^{p} \sum_{m=1}^{p} \sum_{n=1}^{p} \frac{\partial^3 \tilde{B}_{ij}^{(s)}}{\partial \tilde{X}_\ell \partial \tilde{X}_m \partial \tilde{X}_n}\Bigg|_{\underline{X}_o} \Delta X_\ell \Delta X_m \Delta X_n + \ldots \tag{17}$$

In terms of index notation,

$$\Delta B_{ij}^{(s)} = \bar{\alpha}_{ij\ell} \Delta X_\ell + \bar{\beta}_{ij\ell m} \Delta X_\ell \Delta X_m + \bar{\gamma}_{ij\ell mn} \Delta X_\ell \Delta X_m \Delta X_n + \ldots \tag{18}$$

where the $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are the partial derivatives of $\tilde{B}_{ij}^{(s)}$ evaluated at $\underline{X}_o$. Therefore, $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$ are entirely deterministic.

In a similar manner, the strains can be related to the stresses. For no initial stresses, the relationship between the stress and strain vector in element s is

$$\tilde{T}^{(s)} = \tilde{C}^{(s)} \tilde{\varepsilon}^{(s)} \tag{19}$$

where $\tilde{C}^{(s)}$ is the elasticity matrix. Since $\tilde{C}^{(s)}$ is generally stochastic

$$\tilde{C}^{(s)} = C_o^{(s)} + \Delta C^{(s)} \tag{20}$$

where $C_o^{(s)}$ is the deterministic elasticity matrix. As in the calculation of ΔK and $\Delta B^{(s)}$, the $\Delta C^{(s)}$ matrix will be computed by expanding $\tilde{C}^{(s)}$ in a Taylor series about the deterministic solution. Thus

$$\Delta C_{ij}^{(s)} = \bar{\bar{\alpha}}_{ij\ell} \Delta X_\ell + \bar{\bar{\beta}}_{ij\ell m} \Delta X_\ell \Delta X_m + \bar{\bar{\gamma}}_{ij\ell mn} \Delta X_\ell \Delta X_m \delta X_n + \ldots \tag{21}$$

where $\bar{\bar{\alpha}}$, $\bar{\bar{\beta}}$ and $\bar{\bar{\gamma}}$ are the partial derivatives of $\tilde{C}_{ij}^{(s)}$ evaluated at $\underline{X}_o$ and are entirely deterministic. Obviously a relationship exists between the matrices ΔK_{ij}, $\Delta B_{ij}^{(s)}$ and $\Delta C_{ij}^{(s)}$ depending on the nature of the stochastic state vector $\underline{X}_o$.

Using the equations developed in this section, expressions are now available for the computation of the structure's random displacements $\tilde{U}$, element strains $\tilde{\varepsilon}^{(s)}$, and stresses $\tau^{(s)}$ entirely in terms of the random load vector ΔF and the vector $\Delta \underline{X}$ of the random variables in the stiffness matrix. Obviously, the amount of computational effort depends on how many terms are retained in equation (13) and to what order are powers of $\Delta \underline{X}$ retained in ΔK [equation (3)], $\Delta B^{(s)}$ [equation (18)] and $\Delta C^{(s)}$ [equation (21)].

2.0 Truncation of Stochastic Displacement Vector in Terms of Powers of K_o^{-1}

Equation (13) expressed the total random displacement vector as

$$\tilde{U} = U_o + \Delta U_1 + \Delta U_2 + \Delta U_3 + \dots \tag{22}$$

where ΔU_i involves the i-th power of the deterministic matrix K_o^{-1}. Successive terms for ΔU_i are related by the factor $K_o^{-1}\Delta K$. As shown by equation (14)

$$\Delta U_i = -(K_o^{-1}\Delta K)\Delta U_{i-1} \ . \tag{23}$$

From the magnitudes of K_o^{-1} and ΔK encountered in engineering problems, it is expected that ΔU_i will be small compared to ΔU_{i-1}. Thus, at this stage of the formulation, it seems reasonable to take only the first two terms to approximate $\tilde{U}$, i.e.,

$$\tilde{U} \cong U_o + \Delta U_1 \tag{24}$$

where ΔU_1 is given by equation (11).

3.0 Truncation of Power Series Expansions for ΔK, ΔB, and ΔC

The expressions for ΔK, $\Delta B^{(s)}$ and $\Delta C^{(s)}$ were developed in terms of a Taylor series about the deterministic state. These are given up to terms of third order by equations (4), (18) and (21). The coefficients of the powers of ΔX must be evaluated by taking the partial derivatives of the stiffness, strain-displacement, and elasticity matrices.

For the present time we will confine the analysis by retaining only the linear terms in ΔX. This is a reasonable assumption if the variances of the random variables in the stiffness matrix remain small. Hence only compution of the first, and not the higher partial derivatives of the $\tilde{K}$, $\tilde{B}^{(s)}$ and $\tilde{C}^{(s)}$ matrices is required. Under this assumption equations (4), (18) and (21) reduce to

$$\Delta K_{ij} = \alpha_{ij\ell} \Delta X_\ell$$

$$\Delta B_{ij}^{(s)} = \bar{\alpha}_{ij\ell} \Delta X_\ell \tag{25}$$

$$\Delta C_{ij}^{(s)} = \bar{\bar{\alpha}}_{ij\ell} \Delta X_\ell$$

4.0 Summary of Structural Response Equations

Under the assumptions made in the previous two sections, the dependent stochastic displacement, strain, and stress vectors may be expressed as

$$\tilde{U}_i = U_{oi} + K_{oij}^{-1}\Delta F_j - K_{oiq}^{-1}\alpha_{qj\ell}U_{oj}\Delta X_\ell \tag{26a}$$

$$\tilde{\varepsilon}_i^{(s)} = [B_{oij}^{(s)} + \bar{\alpha}_{ij\ell}\Delta X_\ell]\tilde{U}_j \tag{26b}$$

$$\tilde{T}_i^{(s)} = [C_{oij}^{(s)} + \bar{\bar{\alpha}}_{ij\ell}\Delta X_\ell]\tilde{\varepsilon}_j^{(s)} \tag{26c}$$

In terms of the random variables ΔF and $\Delta \underline{X}$, the strain and stresses may be finally expressed as

$$\tilde{\varepsilon}_i^{(s)} = [B_{oij}^{(s)} + \bar{\alpha}_{ij\ell}\Delta X_\ell] \cdot [U_{oj} + K_{ojn}^{-1}\Delta F_n - K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_r] \tag{27}$$

$$\tilde{T}_i^{(s)} = [C_{oip}^{(s)} + \bar{\bar{\alpha}}_{ip\ell}\Delta X_\ell] \cdot [B_{opj}^{(s)} + \alpha_{pjm}\Delta X_m] \cdot [U_{oj} + K_{ojn}^{-1}\Delta F_n - K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_r] \tag{28}$$

Equations (26a), (27) and (28) formulate the global stochastic structural and element response entirely in terms of the random loads and variables in the stiffness matrix. Note that the displacement vector $\tilde{U}$ is linear in the random variables, while the strains and stresses contain terms up to quadratic and cubic in ΔF and $\Delta \underline{X}$, respectively.

This formulation directly reduces to the deterministic solution if no random variable exist in the loading or structural stiffness. That is

$$\bar{U}_i = U_{oi}$$

$$\bar{\varepsilon}_i^{(s)} = B_{oij}^{(s)} U_{oj} \qquad (29)$$

$$\bar{T}_i^{(s)} = C_{oip}^{(s)} B_{opj}^{(s)} U_{oj}$$

The next section applies this formulation to the specific problem of a three-bar truss.

5.0 Example Applied to a Three-Bar Truss

To demonstrate the procedures, let us consider the three-bar truss shown in Figure 1. The bars can take only axial loads, and displacements of the loaded end remain in the X-Y plane.

The stochastic matrix equation for the system is

$$\tilde{K}\tilde{U} = \tilde{F} \qquad (30)$$

where

$$\tilde{U} = \begin{bmatrix} \tilde{U}_x \\ \tilde{U}_y \end{bmatrix} \qquad \tilde{F} = \begin{bmatrix} \tilde{F}_x \\ \tilde{F}_y \end{bmatrix} \qquad (31)$$

The stiffness 2 x 2 matrix for the system is given by

$$\tilde{K}_{rs} = \begin{bmatrix} \sum_{i=1}^{3} \frac{\tilde{A}_i \tilde{E}_i}{\tilde{\ell}_i} \cos^2 \tilde{\theta}_i & \sum_{i=1}^{3} \frac{\tilde{A}_i \tilde{E}_i}{\tilde{\ell}_i} \cos \tilde{\theta}_i \sin \tilde{\theta}_i \\ \text{Symmetric} & \sum_{i=1}^{3} \frac{\tilde{A}_i \tilde{E}_i}{\tilde{\ell}_i} \sin^2 \tilde{\theta}_i \end{bmatrix} \qquad (32)$$

In the general stochastic finite element problem the cross-sectional areas ($\tilde{A}_i$), modulii of elasticity ($\tilde{E}_i$), bars' lengths ($\tilde{\ell}_i$) and load vector $\tilde{F}$, may all be considered random. However, the bars' lengths ($\tilde{\ell}_i$) and the angles ($\tilde{\theta}_i$) (see Figure 1) must satisfy geometric compatibility conditions even though they are random.

In this example, we will somewhat simplify the problem by considering that in the undeformed state, the truss is geometrically symmetric. Thus

$$\begin{aligned} \tilde{\theta}_1 &= -\tilde{\theta}_3 = \tilde{\theta} \\ \tilde{\theta}_2 &= 0 \end{aligned} \tag{33}$$

The lengths of the bars may be random. However, the lengths of bar 1 and 3 are related to the length of bar 2 by the compatibility relationship

$$\tilde{\ell}_1 = \tilde{\ell}_3 = \frac{\tilde{\ell}_2}{\cos \tilde{\theta}} \tag{34}$$

The length of bar 2 (denoted now as $\tilde{\ell}$) can also be expressed in terms of the fixed distance d and the angle $\tilde{\theta}$ as

$$\tilde{\ell} = d \cot \tilde{\theta} \tag{35}$$

The stiffness matrix may now be expressed as a function of the seven random variables $\tilde{A}_i$, $\tilde{E}_i$, and $\tilde{\ell}$

$$\tilde{K}_{rs} = \begin{bmatrix} (\tilde{A}_1\tilde{E}_1 + \tilde{A}_3\tilde{E}_3) \dfrac{\tilde{\ell}^2}{(\tilde{\ell}^2 + d^2)^{3/2}} + \dfrac{\tilde{A}_2\tilde{E}_2}{\tilde{\ell}} & \dfrac{(\tilde{A}_1\tilde{E}_1 - \tilde{A}_3\tilde{E}_3)\, \tilde{\ell} d}{(\tilde{\ell}^2 + d^2)^{3/2}} \\ \text{Symmetric} & \dfrac{(\tilde{A}_1\tilde{E}_1 + \tilde{A}_3\tilde{E}_3) d^2}{(\tilde{\ell}^2 + d^2)^{3/2}} \end{bmatrix} \tag{36}$$

We will now proceed to calculate the ΔK stiffness matrix by the Taylor series expansion about the deterministic state. However, at this stage since we will retain only linear terms, only the first partial derivatives need be computed. Under this assumption

$$\Delta K_{rs} \doteq \alpha_{rs\ell} \Delta X_\ell \tag{37}$$

The terms $\alpha_{rs\ell}$ are evaluated from the partial derivatives of $\bar{K}_{rs}$. These partials evaluated at the determinstic solution are, after defining $\bar{\ell}_o = (\ell_o^2 + d^2)^{\frac{1}{2}}$

$$\frac{\partial \bar{K}_{11}}{\partial A_1} = \frac{E_{o1}\ell_o^2}{\bar{\ell}_o^3} \qquad \frac{\partial \bar{K}_{11}}{\partial E_1} = \frac{A_{o1}\ell_o^2}{\bar{\ell}_o^3}$$

$$\frac{\partial \bar{K}_{11}}{\partial A_2} = \frac{E_{o2}}{\ell_o} \qquad \frac{\partial \bar{K}_{11}}{\partial E_2} = \frac{A_{o2}}{\ell_o} \tag{38}$$

$$\frac{\partial \bar{K}_{11}}{\partial A_3} = \frac{E_{o3}\ell_o^2}{\bar{\ell}_o^3} \qquad \frac{\partial \bar{K}_{11}}{\partial E_3} = \frac{A_{o3}\ell_o^2}{\bar{\ell}_o^3}$$

$$\frac{\partial \bar{K}_{11}}{\partial \ell} = \frac{(A_{o1}E_{o1} + A_{o3}E_{o3})(2\ell_o d^2 - \ell_o^3)}{\bar{\ell}_o^5} - \frac{A_{o2}E_{o2}}{\ell_o^2}$$

$$\frac{\partial \bar{K}_{12}}{\partial A_1} = \frac{\partial \bar{K}_{21}}{\partial A_1} = \frac{E_{o1}\ell_o d}{\bar{\ell}_o^3} \qquad \frac{\partial \bar{K}_{12}}{\partial E_1} = \frac{\partial \bar{K}_{21}}{\partial E_1} = \frac{A_{o1}\ell_o d}{\bar{\ell}_o^3}$$

$$\frac{\partial \bar{K}_{12}}{\partial A_2} = \frac{\partial \bar{K}_{21}}{\partial A_2} = 0 \qquad \frac{\partial \bar{K}_{12}}{\partial E_2} = \frac{\partial \bar{K}_{21}}{\partial E_2} = 0 \tag{39}$$

$$\frac{\partial \bar{K}_{12}}{\partial A_3} = \frac{\partial \bar{K}_{21}}{\partial A_3} = -\frac{E_{o3}\ell_o d}{\bar{\ell}_o^3} \qquad \frac{\partial \bar{K}_{12}}{\partial E_3} = \frac{\partial \bar{K}_{21}}{\partial E_3} = -\frac{A_{o3}\ell_o d}{\bar{\ell}_o^3}$$

$$\frac{\partial \bar{K}_{12}}{\partial \ell} = \frac{\partial \bar{K}_{21}}{\partial \ell} = \frac{(A_{o1}E_{o1} - A_{o3}E_{o3})(d^3 - 2\ell_o^2 d)}{\bar{\ell}_o^5}$$

$$\frac{\partial \tilde{K}_{33}}{\partial A_1} = \frac{E_{o1} d^2}{\bar{\ell}_o^3} \qquad \frac{\partial \tilde{K}_{33}}{\partial E_1} = \frac{A_{o1} d^2}{\bar{\ell}_o^3}$$

$$\frac{\partial \tilde{K}_{33}}{\partial A_2} = 0 \qquad \frac{\partial \tilde{K}_{33}}{\partial E_2} = 0 \tag{40}$$

$$\frac{\partial \tilde{K}_{33}}{\partial A_3} = \frac{E_{o3} d^2}{\bar{\ell}_o^3} \qquad \frac{\partial \tilde{K}_{33}}{\partial E_3} = \frac{A_{o3} d^2}{\bar{\ell}_o^3}$$

$$\frac{\partial \tilde{K}_{33}}{\partial \ell} = - \frac{3(A_{o1}E_{o1} + A_{o3}E_{o3})d^2 \ell_o}{\bar{\ell}_o^5}$$

Let us define the vector of random variables, in the stiffness matrix as measured from the deterministic solution,

$$\Delta \underline{X} = \tilde{\underline{X}} - \underline{X}_o = \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \Delta X_3 \\ \Delta X_4 \\ \Delta X_5 \\ \Delta X_6 \\ \Delta X_7 \end{bmatrix} = \begin{bmatrix} \Delta A_1 \\ \Delta A_2 \\ \Delta A_3 \\ \Delta E_1 \\ \Delta E_2 \\ \Delta E_3 \\ \Delta \ell \end{bmatrix} \tag{41}$$

Specific values of $\alpha_{rs\ell}$ now become

$$\alpha_{rs1} = \frac{\partial \bar{K}_{rs}}{\partial A_1} \qquad \alpha_{rs4} = \frac{\partial \bar{K}_{rs}}{\partial E_1} \qquad \alpha_{rs7} = \frac{\partial \bar{K}_{rs}}{\ell}$$

$$\alpha_{rs2} = \frac{\partial \bar{K}_{rs}}{\partial A_2} \qquad \alpha_{rs5} = \frac{\partial \bar{K}_{rs}}{\partial E_2} \tag{42}$$

$$\alpha_{rs3} = \frac{\partial \bar{K}_{rs}}{\partial A_3} \qquad \alpha_{rs6} = \frac{\partial \bar{K}_{rs}}{\partial E_3}$$

where $\frac{\partial \bar{K}_{rs}}{\partial X_\ell}$ are evaluated from equations (38-40).

In a linear elastic finite element formulation, the first step in evaluating axial strains is to compute the component of displacement projected on the original positions of the bars and dividing this displacement by the original length. Thus

$$\text{Beam 1} \quad \tilde{\varepsilon}^{(1)} = \frac{\cos\tilde{\theta}\,\bar{U}_x + \sin\tilde{\theta}\,\bar{U}_y}{\tilde{\ell}_1}$$

$$= \frac{\cos^2\tilde{\theta}\,\bar{U}_x + \sin\tilde{\theta}\cos\tilde{\theta}\,\bar{U}_y}{\tilde{\ell}} \tag{43}$$

$$= \frac{\tilde{\ell}\,\bar{U}_x + d\,\bar{U}_y}{(\tilde{\ell}^2 + d^2)}$$

$$\text{Beam 2} \quad \tilde{\varepsilon}^{(2)} = \frac{\bar{U}_x}{\tilde{\ell}} \tag{44}$$

$$\text{Beam 3} \quad \tilde{\varepsilon}^{(3)} = \frac{\cos\tilde{\theta}\,\bar{U}_x - \sin\tilde{\theta}\,\bar{U}_y}{\tilde{\ell}_3}$$

$$= \frac{\tilde{\ell}\,\bar{U}_x - d\,\bar{U}_y}{(\tilde{\ell}^2 + d^2)} \tag{45}$$

The strain displacement matrix $\tilde{B}$ relates the element strains to the nodal displacements.

$$\tilde{\varepsilon}^{(s)} = \tilde{B}^{(s)} \tilde{U} \tag{46}$$

In this case

$$\tilde{B}^{(s)} = \begin{bmatrix} \dfrac{\tilde{\ell}}{\tilde{\ell}^2 + d^2} & \dfrac{d}{\tilde{\ell}^2 + d^2} \\ \dfrac{1}{\tilde{\ell}} & 0 \\ \dfrac{\tilde{\ell}}{\tilde{\ell}^2 + d^2} & \dfrac{-\ d}{\tilde{\ell}^2 + d^2} \end{bmatrix} \tag{47}$$

In previous discussions the stochastic $\tilde{B}^{(s)}$ was separated into deterministic and stochastic parts

$$\tilde{B}^{(s)} = B_o^{(s)} + \Delta B^{(s)} \tag{48}$$

The deterministic part $B_o^{(s)}$ can be evaluated by substituting ℓ_o for $\tilde{\ell}$ in equation (47). $\Delta B^{(s)}$ can be computed by expanding $\tilde{B}^{(s)}$ in a Taylor series about ℓ_o. To the first order terms in $\Delta\ell(\Delta X_7)$ this gives

$$\Delta B^{(s)} = \begin{bmatrix} \dfrac{(d^2 - \ell_o^2)\Delta X_7}{\ell_o^4} & \dfrac{2\ell_o d\Delta X_7}{\ell_o^4} \\ -\dfrac{\Delta X_7}{\ell_o^2} & 0 \\ \dfrac{(d^2 - \ell_o^2)\Delta X_7}{\ell_o^4} & -\dfrac{2\ell_o d\Delta X_7}{\ell_o^4} \end{bmatrix} \tag{49}$$

In index notation $\Delta B_{ij}^{(s)}$ was expressed for element s as

$$\Delta B_{ij}^{(s)} = \bar{\alpha}_{ij\ell} \Delta X_{\ell} \tag{50}$$

$\bar{\alpha}_{ij\ell}$ can be evaluated from expression (49). Thus $\bar{\alpha}_{ij\ell} = 0$ for all $ij\ell$ except

$$\begin{aligned} \bar{\alpha}_{117} &= \bar{\alpha}_{317} = \frac{\left(d^2 - \ell_o^2\right)}{\bar{\ell}_o^4} \\ \bar{\alpha}_{127} &= - \bar{\alpha}_{317} = \frac{2\ell_o d}{\bar{\ell}_o^4} \\ \bar{\alpha}_{217} &= - \frac{1}{\ell_o^2} \end{aligned} \tag{51}$$

To complete the calculation of the dependent random variables, the expressions for the stochastic stresses must be derived. In general, stresses are calculated from

$$\tilde{\sigma}^{(s)} = \tilde{C}^{(s)} \varepsilon^{(s)} \tag{52}$$

where $\tilde{C}^{(s)}$ is the elasticity matrix. For the three-bar truss problem, the $\tilde{C}^{(s)}$ matrix only involves the modulus of elasticity. Thus, following the usual notation

$$\tilde{C} = C_o + \Delta C = \begin{bmatrix} E_{o1} & 0 & 0 \\ 0 & E_{o2} & 0 \\ 0 & 0 & E_{o3} \end{bmatrix} + \begin{bmatrix} \Delta E_1 & 0 & 0 \\ 0 & \Delta E_2 & 0 \\ 0 & 0 & \Delta E_3 \end{bmatrix} \tag{53}$$

The Taylor series expansion about the deterministic state only yields terms to first order since the elasticity matrix is linear in modulus.

In index notation ΔC is

$$\Delta C_{ij}^{(s)} = \bar{\bar{\alpha}}_{ij\ell} \Delta X_\ell \tag{54}$$

where $\Delta X_4 = \Delta E_1$

$\Delta X_5 = \Delta E_2$

$\Delta X_6 = \Delta E_3$

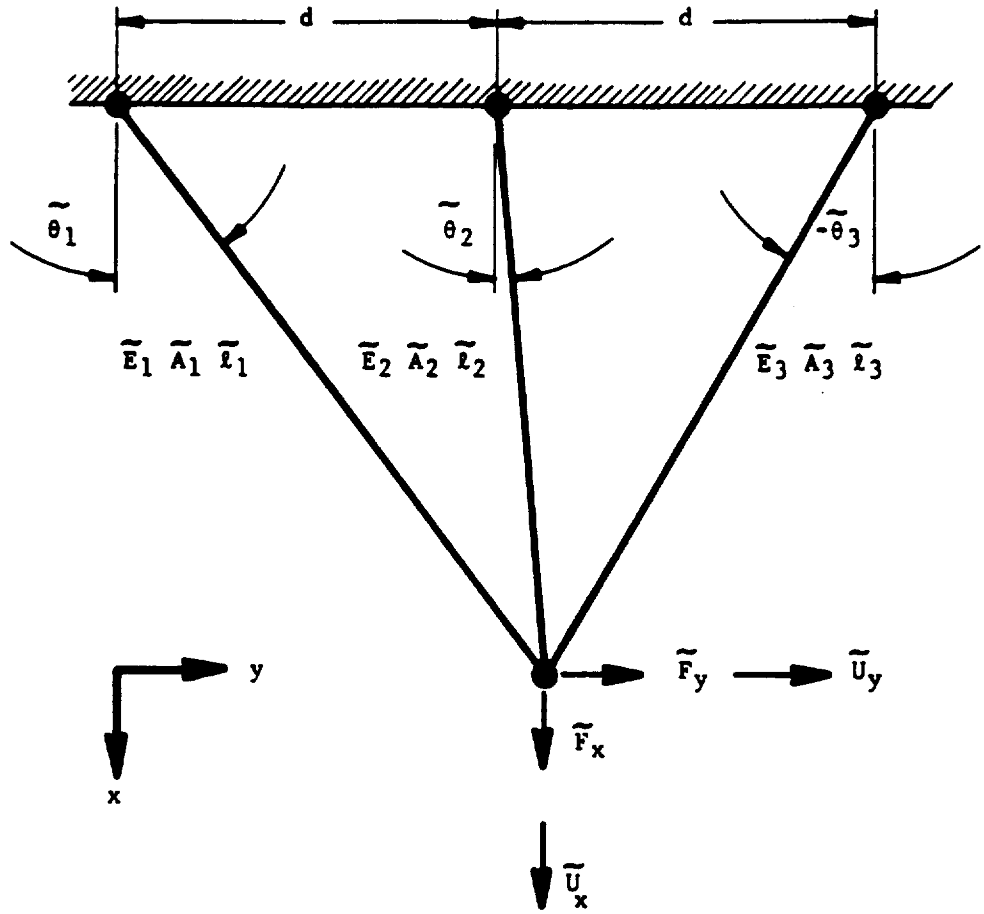
This gives for $\bar{\bar{\alpha}}_{ij\ell}$

$$\bar{\bar{\alpha}}_{ij\ell} = 0 \text{ for all } ij\ell \text{ except} \tag{55}$$

$$\bar{\bar{\alpha}}_{114} = \bar{\bar{\alpha}}_{225} = \bar{\bar{\alpha}}_{336} = 1$$

This completes the formulation for all quantities used in equations (26a), (27) and (28) for calculation of the stochastic displacements, strains, and stresses. The next issue concerns what probabilistic methods can be employed to evaluate the probabilistic response of the system.

In closing, a few comments on the question of geometric compatibility are in order. In this problem, the general stiffness matrix [equation (32)] is given in terms of the angles $\tilde{\theta}_i$ and lengths $\tilde{\ell}_i$. The angles $\tilde{\theta}_i$ were eliminated in the formulation through compatibility conditions, and the final results only contained the random variable $\Delta\ell(\Delta X_7)$. An alternate approach is to retain the random variables $\Delta\theta_i$ and $\Delta\ell_i$ in the formulation of $\tilde{U}$, $\tilde{\varepsilon}^{(s)}$, and $\tilde{T}^{(s)}$ in equations (26a), (27) and (28). The compatibility condition could then be imposed through the correlation coefficients between the random variables for the probabilistic evaluation.



Three-Bar Truss

FIGURE 1

6.0 Expected Values, Variances and Covariances of the Stochastic Displacement Vector

The stochastic displacement vector $\tilde{U}_i$ was shown in Section 4.0 to be a linear combination of the stochastic force vector and the random variables in the stiffness matrix. From equation (26a)

$$\tilde{U}_i = U_{oi} + K^{-1}_{oij}\Delta F_j - K^{-1}_{oiq}\alpha_{qj\ell}U_{oj}\Delta X_\ell \tag{56}$$

Let us define another another random vector $\Delta \underline{Y}$ as containing both the ΔF and $\Delta \underline{X}$ vectors, i.e.,

$$\Delta \underline{Y} = \begin{bmatrix} \Delta F \\ \\ \Delta \underline{X} \end{bmatrix} \tag{57}$$

The random displacement vector $\tilde{U}_i$ may now be written as

$$\tilde{U}_i = U_{oi} + \alpha^*_{ij}\Delta Y_j \tag{58}$$

where

$$\alpha^*_{ij} = \begin{bmatrix} K^{-1}_{oij} & 0 \\ \\ 0 & -K^{-1}_{oiq}\alpha_{q\ell j}U_{o\ell} \end{bmatrix}$$

The mean or expected value of $\tilde{U}_i$ is

$$\begin{aligned} E(\tilde{U}_i) &= E(U_{oi} + \alpha^*_{ij}\Delta Y_j) \\ &= E(U_{oi}) + E(\alpha^*_{ij}\Delta Y_j) \\ &= U_{oi} + \alpha^*_{ij}E(\Delta Y_j) \end{aligned} \tag{59}$$

The variance of $\tilde{U}_i$ is

$$V(\tilde{U}_i) = E\{[\tilde{U}_i - E(\tilde{U}_i)]^2\}$$
$$= E(\tilde{U}_i^{\,2}) - [E(\tilde{U}_i)]^2 \qquad (60)$$

which for equation (58) reduces to

$$V(\tilde{U}_i) = \sum_{j=1}^{n} (\alpha^*_{ij})^2 \, V(\Delta Y_j)$$
$$+ \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{k=1}^{n} \alpha^*_{ij}\alpha^*_{ik} \, Cov(\Delta Y_j, \Delta Y_k) \qquad (61)$$

The covariance term $Cov(\Delta Y_j \Delta Y_k)$, denoted as ρ_{jk}, is defined as

$$Cov(\Delta Y_j \Delta Y_k) = E[\{\Delta Y_j - E(\Delta Y_j)\}\{\Delta Y_k - E(\Delta Y_k)\}]$$
$$= E(\Delta Y_j \Delta Y_k) - E(\Delta Y_j)E(\Delta Y_k) \qquad (62)$$

If the random variables in $\Delta \underline{Y}$ are independent, then they are uncorrelated and

$$Cov(\Delta Y_j, \Delta Y_k) = \rho_{jk} = 0 \qquad (63)$$

In this case, the variance of $\tilde{U}_i$ becomes simply

$$V(\tilde{U}_i) = \sum_{j=1}^{n} (\alpha^*_{ij})^2 \, V(\Delta Y_j) \qquad (64)$$

The correlation coefficient will not be zero in the general case and must be evaluated. Figure 2 illustrates how correlation could occur within each set of random vectors ΔF and $\Delta \underline{X}$ and even between ΔF and $\Delta \underline{X}$.

7.0 Higher Moments and the Distribution of $\tilde{U}_i$ for Independent Random Variables

The special case of independent, i.e., uncorrelated random variables permits a relatively simple calculation for the higher-order moments of the stochastic variable $\tilde{U}_i$. Hines and Montgomery [2]* show that if $M_{\Delta Y_j}(t)$ is the moment generating function of ΔY_j, then the moment generating function of $\tilde{U}_i$ for equation (58) is

$$M_{\tilde{U}_i}(t) = e^{U_{oi}t}\left[M_{\Delta Y_1}(\alpha^*_{i1}t)\cdot M_{\Delta Y_2}(\alpha^*_{i2}t)\cdot \ldots \cdot M_{\Delta Y_n}(\alpha^*_{in}t)\right] \tag{65}$$

Recall that the moment generating function for the random variable Z is defined as

$$M_Z(t) = E(e^{tZ}) \tag{66}$$

and has the property

$$\frac{d^r M_Z(t)}{dt^r} = E(Z^r e^{tZ}) \tag{67}$$

Thus

$$E(Z^r) = \left.\frac{d^r M_Z(t)}{dt^r}\right|_{t=0} \tag{68}$$

From knowledge of the type distributions of the <u>independent</u> random variables in $\Delta \underline{Y}$, we know their moment generating functions. From equation (65), the moment generating function for $\tilde{U}_i$ can be constructed, and the r-th moment is

$$m_r = E(\tilde{U}_i^r) = \left.\frac{d^r}{dt^r} M_{\tilde{U}_i}(t)\right|_{t=0} \tag{69}$$

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[2]* <u>Probability of Statistics in Engineering Management and Science</u>, Wiley, 1980.

The probability density function (pdf) $p_{\tilde{U}_i}(u_i)$ is related to the r-th moment by

$$E(\tilde{U}_i^k) = \int_{-\infty}^{\infty} u_i^k p_{\tilde{U}_i}(u_i)du_i \tag{70}$$

In theory, at least, from a knowledge of $E(\tilde{U}_i^k)$, we can construct the probability density function $p_{\tilde{U}_i}(u_i)$. This would allow computation of the cumulative distribution function (cdf) $P_{\tilde{U}_i}(u_i)$, where the probability that $\tilde{U}_i \le u_i$ is defined as

$$P(\tilde{U}_i \le u_i) \equiv P_{\tilde{U}_i}(u_i) \tag{71}$$

$$= \int_{-\infty}^{u_i} p_{\tilde{U}_i}(u_i)du_i$$

8.0 Expected Values, Variances and Covariances of the Stochastic Strain Vector

From equation (27) in Section 4.0, the stochastic strain vector was expressed in the form

$$\tilde{\varepsilon}_i^{(s)} = [B_{oij}^{(s)} + \bar{\alpha}_{ij\ell}\Delta X_\ell]\cdot[U_{oj} + K_{ojn}^{-1}\Delta F_n - K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_r] \tag{72}$$

Expanding the expression gives

$$\begin{aligned}\tilde{\varepsilon}_i^{(s)} = {} & B_{oij}^{(s)}U_{oj} + B_{oij}^{(s)}K_{ojn}^{-1}\Delta F_n - B_{oij}^{(s)}K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_r \\ & + \bar{\alpha}_{ij\ell}U_{oj}\Delta X_\ell + \bar{\alpha}_{ij\ell}K_{ojn}^{-1}\Delta X_\ell\Delta F_n - \bar{\alpha}_{ij\ell}K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_\ell\Delta X_r\end{aligned} \tag{73}$$

If the product terms in the random variables are neglected in equation (73), i.e.,

$$\Delta X_{\ell} \Delta F_{n} \approx 0$$

and (74)

$$\Delta X_{\ell} \Delta X_{r} \approx 0$$

then the strain $\tilde{\varepsilon}_i^{(s)}$ is reduced to a linear form. Using the vector $\Delta \underline{Y}$ defined by equation (57), the expression (73) for strain may be written as

$$\tilde{\varepsilon}_i^{(s)} = \varepsilon_{oi}^{(s)} + \bar{\alpha}_{ij}^{*} \Delta Y_j \tag{75}$$

where

$$\varepsilon_{oi}^{(s)} = B_{oij}^{(s)} U_{oj}$$

$$\bar{\alpha}_{ij}^{*} = \begin{bmatrix} B_{oi\ell}^{(s)} K_{o\ell j}^{-1} & 0 \\ & \\ 0 & \begin{matrix} -B_{oi\ell}^{(s)} K_{o\ell q}^{-1} \alpha_{qnj} U_{on} \\ + \bar{\alpha}_{i\ell j} U_{o\ell} \end{matrix} \end{bmatrix} \tag{76}$$

The stochastic strains in equation (75) are of the same form as equation (58) for stochastic displacements $\tilde{U}_i$. Therefore, the development of the expected values, moments, and distributions of $\tilde{\varepsilon}_i^{(s)}$ follow along the lines given in Sections 6.0 and 7.0.

9.0 Expected Values, Variances and Covariances of the Stochastic Stress Vector

The same linearization strategy will be followed with the stochastic stress vector. From equation (28) the stress vector was expressed as

$$\tilde{T}_i^{(s)} = \left[C_{oip}^{(s)} + \bar{\bar{\alpha}}_{ip\ell}\Delta X_\ell\right]\cdot\left[B_{opj}^{(s)} + \bar{\alpha}_{pjm}\Delta X_m\right]\cdot$$

$$\left[U_{oj} + K_{ojn}^{-1}\Delta F_n - K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_r\right] \qquad (77)$$

Expanding equation (77) and retaining only linear terms gives

$$\tilde{T}_i^{(s)} = C_{oip}^{(s)}B_{opj}^{(s)}U_{oj} + C_{oip}^{(s)}B_{opj}^{(s)}K_{ojn}^{-1}\Delta F_n$$

$$- C_{oip}^{(s)}B_{opj}^{(s)}K_{ojq}^{-1}\alpha_{qnr}U_{on}\Delta X_r \qquad (78)$$

$$+ \bar{\bar{\alpha}}_{ip\ell}B_{opj}^{(s)}U_{oj}\Delta X_\ell + C_{oip}^{(s)}\bar{\alpha}_{pjm}U_{oj}\Delta X_m$$

This expression for $\tilde{T}_i^{(s)}$ may be written in the simplified form

$$\tilde{T}_i^{(s)} = \tilde{T}_{oi}^{(s)} + \bar{\bar{\alpha}}_{ij}^{*}\Delta Y_j \qquad (79)$$

where

$$\Delta Y = \begin{bmatrix} \Delta F \\ \\ \Delta \underline{X} \end{bmatrix} \qquad (80)$$

$$T^{(s)}_{oi} = C^{(s)}_{oip} B^{(s)}_{opj} U_{oj}$$

$$\bar{\bar{\alpha}}^{*}_{ij} = \begin{bmatrix} C_{oip} B_{op\ell} K^{-1}_{o\ell j} & 0 \\ 0 & \begin{matrix} - C^{(s)}_{oip} B_{op\ell} K^{-1}_{o\ell q} \alpha_{qnj} U_{on} \\ + C^{(s)}_{oip} \bar{\alpha}_{p\ell j} U_{o\ell} \\ + \bar{\bar{\alpha}}_{ipj} B^{(s)}_{op\ell} U_{o\ell} \end{matrix} \end{bmatrix} \tag{81}$$

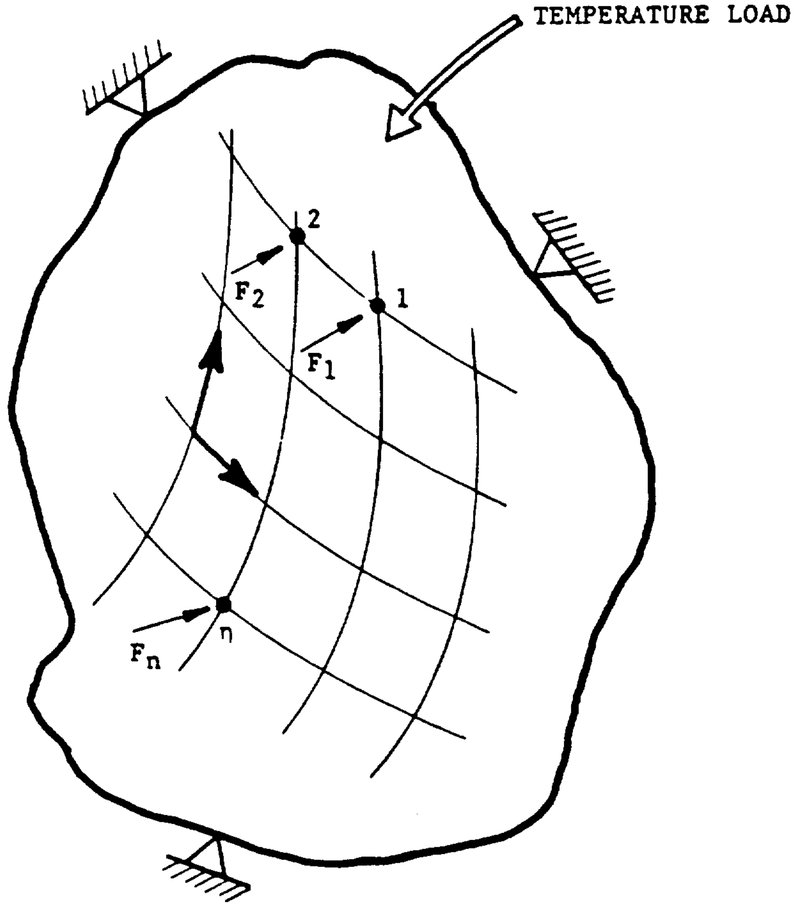
The linear form of equation (79) allows the same development for expected values, moments, and distributions of $\tilde{T}^{(s)}_i$ as for $\tilde{U}_i$ and $\tilde{\varepsilon}^{(s)}_i$ in Sections 6.0 thru 8.0.

10.0 Comments on Implications of Linear Formulation

In the previous sections the dependent stochastic variables in displacement, strain, and stress were cast in the linear format

$$\begin{aligned} \tilde{U}_i &= \tilde{U}_{oi} + \bar{\alpha}^{*}_{ij} \Delta Y_j \\ \tilde{\varepsilon}^{(s)}_i &= \tilde{\varepsilon}^{(s)}_{oi} + \bar{\alpha}^{*}_{ij} \Delta Y_j \\ \tilde{T}^{(s)}_i &= \tilde{T}^{(s)}_{oi} + \bar{\bar{\alpha}}^{*}_{ij} \Delta Y_j \end{aligned} \tag{82}$$

First, such a format has several nice properties in the evaluation of the probabilistic response. If the elements of the random variable vector $\Delta \underline{Y}$ are each <u>normal</u> and <u>independent</u>, then the random variables vectors $\tilde{U}^{(s)}_i$, $\tilde{\varepsilon}^{(s)}_i$ and $\tilde{T}^{(s)}_i$ are normal. Furthermore, if the elements of ΔY_j are <u>independent</u>, then $\tilde{U}^{(s)}_i$, $\tilde{\varepsilon}^{(s)}_i$ and $\tilde{T}^{(s)}_i$ approach normal distributions as the size of the $\Delta \underline{Y}$ vector, i.e., dimension j, becomes large. This is irrespective of the type of distributions in the $\Delta \underline{Y}$ vector.



- o Correlation Between Material Properties as a Function of Location
- o Correlation Between Applied Forces
- o Correlation Between Material Properties and Applied Loads, e.g., Temperature Dependent Properties

FIGURE 2. EXAMPLES OF CORRELATION BETWEEN RANDOM VARIABLES

Section 4

A Preliminary Plan for Validation of the First Year PFEM Code

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General Description

This plan describes test cases designed to validate the first-year edition of the PSAM PFEM computer code which has the following capabilities:

1. The code combines the conventional finite element method using the newly developed perturbation scheme for generating analytical performance function, with the fast probability integration (FPI) algorithm. The performance functions will consider all important design-related variables such as the displacements, strains, stresses, natural frequencies, limit states, etc.

2. The code employs complex elements (beams, plane stress/strain, axisymmetric, plates, and shells), linear elastic material behavior, and small deformations.

3. For static problems, the loading vector can be treated as a correlated random vector. For dynamic problems, the steady-state random responses of a linear stochastic structure to stationary random loading can be solved.

4. The output of the code, for probabilistic design purposes, include the probability density functions (pdf) and the cumulative distribution functions (cdf) of the performance functions, and the probability of exceedence or the reliability of the response variables.

The following cases were carefully selected to test the many features of the PFEM code. Most of the cases are considered well-defined, with specified numerical design values and distributional information. However, there are some cases where only exact or good approximating performance function are given. Those are the cases where some trial and error procedures must be done to generate meaningful input data. A case for testing probabilistic shell response has not been finalized yet due to the difficulty of finding a "good" performance function for checking purposes. A test case involving a twisted cantilevered plate (relating closely to the turbine blade in the PSAM project) may be included in the validation, depending on the availability of the accurate theoretical solution of the natural frequencies.

In summary, the primary purpose of this test plan is to define the scope of the PFEM code validation. All test cases, in their final version, will be given an "exact" performance function, so that the advanced reliability analysis methods, as well as Monte Carlo simulations, can be applied for the comparisons. The test cases will, in general, fall within the above-defined capabilities of the PFEM code, and will cover all important aspects of the probabilistic design.

Test Cases

Case 1. Cantilever Beam (Solution is given in Section 4 Appendix)

The cantilever beam modeled in Figure 1 is subjected to static loadings, $P_i(i=1,5)$. P_i's are correlated with the correlation coefficient defined as:

$$\rho_{P_i P_j} = \exp\left[-C\left(\frac{\Delta x_{ij}}{L}\right)\right] \quad (1)$$

where C is a constant, Δx_{ij} is the distance from element i to element j, and L is the length of the beam.

The tip displacement is:

$$\delta = \sum_{i=1}^{5}\left[\frac{2P_i \ell_i^2}{Ewt^3}(3L-\ell_i) + \frac{P_i \ell_i L}{K}\right] \quad (2)$$

where ℓ_i is the distance from the support point A to the point where P_i applies. The "fixed-end" is not exactly fixed, and is modeled using a torsion spring with spring constant K.

The following distributional data is assumed:

P_i ~ Normal (20, .1) lb
E ~ Lognormal (10^7, .03) psi
L ~ Lognormal (20, .05) in.
t ~ Lognormal (0.98, .05) in.
w ~ Lognormal (1.0, .05) in.
K ~ Lognormal (10^5, .05) in-lb/rad
σ_Y ~ Weibull (10^4, .10) psi
ν = 0.3
C = 1

where X Normal (μ_x, C_x) means the variable X has a normal distribution with μ_x = mean value and C_x = coefficient of variation. If the distribution is a lognormal, then μ_x = median.

Another performance function considered is the maximum stress at point A, denoted as S,

$$S = \frac{6\Sigma P_i \ell_i}{wt^2} \quad (3)$$

The correlated loading vector, $\underset{\sim}{P}$, can be transformed to an uncorrelated vector, $\underset{\sim}{p}$, using

$$\underset{\sim}{p} = \underset{\sim}{A}^T \underset{\sim}{P} \tag{4}$$

Where $\underset{\sim}{A}$ is an orthogonal matrix with column vectors equal to the eigenvectors of the covariance matrix, $\underset{\sim}{C}_P$;

$$\underset{\sim}{C}_P = \begin{bmatrix} \sigma_{11}^2 & \cdot & \cdot & \sigma_{1n} \\ \vdots & & & \vdots \\ \sigma_{n1} & \cdot & \cdot & \sigma_{nn}^2 \end{bmatrix} \tag{5}$$

where

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j \tag{6}$$

Using the inverse transformation of Eq. (4), the two performance functions, namely δ and S, may be written in terms of $\underset{\sim}{p}$, for example,

$$\delta = \{\frac{2(3L\ell_i^2 - \ell_i^3)}{Ewt^3} + \frac{\ell_i L}{K}\}^T \underset{\sim}{A}^{T^{-1}} \underset{\sim}{p} \tag{7}$$

where $\{\cdot\}$ is a column vector. The mean and the standard deviation of p can be computed as:

$$\underset{\sim}{\mu}_p = \underset{\sim}{A}^T \underset{\sim}{\mu}_P \tag{8}$$

$$\underset{\sim}{\sigma}_p^2 = \underset{\sim}{\lambda} \tag{9}$$

where $\underset{\sim}{\lambda}$ is a vector of the eigenvalues of $\underset{\sim}{C}_p$. The deterministic solutions, using the mean (or median for lognormal variables) values, are

$$\mu_\delta = 0.5 \text{ in.}$$

and

$$\mu_S = 7497 \text{ psi}$$

which may be used to check the deterministic solutions of the PFEM code.

The expected output of the code include:

1. The pdf and the cdf of the tip displacement, δ.
2. The pdf and the cdf of the maximum stress, S.
3. The probability that the stress S will exceed the yield strength, σ_y.

The above results of the PFEM code will be checked by FPI program and Monte Carlo simulation using the exact performance functions.

Case 2. Cantilever Plate

This test case is for checking the plate element. The problems and the data are the same as described in Case 1, except that the median of the thickness will be taken as 0.1 in.

Case 3. Cantilever Beam (Natural Frequency)

The primary goal of this test case is to test the capability of the perturbation algorithm for the eigenvalue problem. The cantilever beam, as given in test Case 1, will be used; but the end point A will be assumed fixed ($K=\infty$). The performance functions to be tested are the first three bending frequencies which may be approximated as

$$\omega_i = \alpha_i \sqrt{\frac{EI}{\rho w t L^4}} \ , \ i=1,3 \tag{10}$$

where α_i are the constants ($\alpha_1 = 3.52$, $\alpha_2 = 22.4$, $\alpha_3 = 61.7$) and ρ is the mass density defined as

$$\rho \sim \text{Lognormal}(2.5\text{x}10^{-4}, 0.05) \text{ lb-sec}^2/\text{in.}^4$$

note that ω_i in Eq. (10) are the natural frequencies for the vibration motion parallel to yz-plane (see Figure 1). To compute ω_i, for xy-plane motion, the thickness (t) and the width (w) need be switched in Eq. (10).

The values of w and t are chosen very closely in order to test the capability of the code in identifying different modes which have approximately the same natural frequencies.

By substituting the mean or median values into Eq. (10), the deterministic solutions of ω_i can be obtained for checking the code solutions. For example,

ω_i = 497.9 rad/sec, yz plane

= 508.1 rad/sec, xy plane

The expected output of the code is the cdf and the pdf of ω_i for both directions.

Case 4. Rotating Beam (Centrifugal Loading and Stress Stiffening Effects)

The geometry of the beam is given in Figure 2. The tip axial displacement due to centrifugal loading is:

$$u(x=\ell) = \frac{2}{3}\frac{\rho\Omega^2\ell^3}{E} \qquad (11)$$

where the variables are defined as:

Ω = 2400 rad/sec

ρ ~ Lognormal (9.0×10^{-4}, 0.05) lb-sec^2/in^4

ℓ ~ Lognormal (3.844, 0.05) in.

E ~ Lognormal (2.9×10^7, 0.1) psi

It can be shown that the tip displacement is also a lognormal variable which can be expressed by:

$u(x=\ell)$ ~ Lognormal (6.77×10^{-3}, 0.188) in.

Stress stiffening on the bending frequency can be included by using Galerkin's method [1].

The performance function for the first bending frequency can be approximated as

$$\omega = \sqrt{\frac{1.0371Et^2}{\rho \ell^4} + 2.886\Omega^2} \tag{12}$$

where the thickness t is given as

t ~ Lognormal (0.0416, 0.05) in.

For the finite element model, the width, w, is given as

w ~ Lognormal (1.424, 0.05) in.

By substituting Ω value into Eq. (12) and letting

$$H = \frac{1.0371\ Et^2}{\rho l^4} \tag{13}$$

which has a lognormal distribution, Eq. (12) becomes

$$\omega = \sqrt{H + 1.662 \times 10^7} \tag{14}$$

Using Eq. (14), the cdf of $\omega, F_\omega(\omega)$, can be expressed in terms of the cdf of H, $F_H(h)$. Because H is a lognormal variable, it can be shown that

$$\begin{aligned} F_\omega(\omega_o) &= F_H(\omega_o^2 - 1.662 \times 10^7) \\ &= \Phi_H\left(\frac{\ln(\omega_o^2 - 1.662 \times 10^7) - \mu_{\ell nH}}{\sigma_{\ell nH}}\right) \end{aligned} \tag{15}$$

where $\Phi_H(\cdot)$ is the standard normal cdf. Therefore, an exact distribution function of ω can be computed easily.

The PFEM code will be expected to generate pdf and cdf for the tip axial displacement and the first bending frequency. The results will be compared with the exact lognormal distributions.

Case 5. <u>Rotating Beam - Plate Element</u>

Repeat Case 4 using plate element.

Case 6. Twisted Cantilevered Plate

A twisted plate is shown in Figure 3 where a rectangular plate of length a, width b and thickness h is shown clamped at one edge, with the opposite edge pretwisted through an angle ϕ.

Although much more complicated, turbine blade is basically a twisted plate and, therefore, the model shown in Figure 3 would be a good validation test case. However, the vibration of rotating blade, even for the relatively simple configuration considered, have found to be very difficult to analyze by theoretical methods. Widespread disagreement has been found among published results, especially when the aspect ratio (a/b) is relatively small, e.g., one or two.

To clarify the problem, a joint industry/government/university NASA-Lewis research effort was initiated to obtain comprehensive theoretical and experimental results [2]. All of the theoretical methods used were found to be either inapplicable or unreliable for certain ranges of the geometric parameters (a/b, b/h, ϕ).

For a validation test to be meaningful, it is essential that the performance function be accurate. For this reason, this test case will be included only if a suitable performance function can be determined.

Case 7. Plate with Different Correlated Loadings in Different Zones

The plate model considered here has a geometry shown in Figure 4 and is simply supported. The plate is separated into four zones with each zone having a different correlated static loading.

The performance function considered is the center (point c) displacement δ which, for a single point loading P, has the following theoretical expression.

$$\delta = \frac{4\ell^2 P}{\pi^4 D}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{\ell}\sin\frac{n\pi y}{\ell}}{(m^2+n^2)^2}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\right] \qquad (16)$$

For simplicity, define F as a factor representing the series within the brackets in Eq. (16), i.e.,

$$\delta = \frac{4\ell^2}{\pi^4 D} PF \qquad (17)$$

where

$$D = \frac{Et^3}{12(1-\nu^2)} \qquad (18)$$

For n loads, δ is

$$\delta = \frac{4\ell^2}{\pi^4 D} \sum_{i=1}^{n} (PF)_i \tag{19}$$

Now consider a total of sixty-four loads as illustrated in Figure 4. The loads will be divided into four zones. In each zone the loading vector will be denoted by P_i. For example, at zone i (i=1,4),

$$\underset{\sim}{P}_i = \begin{Bmatrix} P_1 \\ \vdots \\ P_{16} \end{Bmatrix}_i \tag{20}$$

The loads in each zone are correlated with the correlation coefficients defined in each zone as

$$\underset{\sim}{\rho}_i = \begin{bmatrix} \rho_{11} & \cdot\ \cdot & \rho_{1,16} \\ \vdots & & \\ \rho_{16,1} & \cdot\ \cdot & \rho_{16,16} \end{bmatrix}_i \tag{21}$$

The elements ρ_{ij} in the above matrix are defined as

$$\rho_{ij} = \exp[-K \sqrt{\Delta x_{ij}^2 + \Delta y_{ij}^2}\,] \tag{22}$$

where Δx and Δy are x-direction and y-direction distances, respectively. Using Eq. (22), a covariance matrix, $\underset{\sim}{C}_i$, can be established to subsequently generate a transformation matrix, $\underset{\sim}{A}_i$, such that

$$\underset{\sim}{p}_i = \underset{\sim}{A}_i^T \underset{\sim}{P}_i \tag{23}$$

where $\underset{\sim}{p}_i$ is the uncorrelated loading vector at zone i.

Using Eq. (23), the middle displacement can be formulated as

$$\delta = \frac{4\ell^2}{\pi^4 D} \sum_{i=1}^{4} \left[(\underset{\sim}{A}^{T^{-1}} \cdot \underset{\sim}{p})^T \underset{\sim}{F} \right]_i \tag{24}$$

which will be used as a comparison basis.

Because of the large amount of random variables involved, the selections of the material properties, as well as the loads and correlation coefficients in the four zones, have not been completed yet. However, the list of the random variables and the deterministic parameters are given as follows:

Random Variables: E, t, ν, ℓ, P_1 to P_{64}

Deterministic Variables: K_1 to K_4

The goal of the analysis is to construct the distribution of δ.

Case 8. Shell Element

Due to the difficulties of selecting a proper performance function for reliability analysis, this case (employing shell elements) has not been determined.

Case 9. Random Vibration [3]

The case chosen is a simply supported beam under concentrated random force F(t) as shown in Figure 5. F(t) is a random function of time, representing band-limited white noise with cutoff frequency ω_c:

$$E[F(t)] = 0$$

$$S_F(\omega) = \begin{cases} S_0, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

where $E[\cdot]$ denotes the mean value, S_F is the spectral density.

The displacement is a stationary random process which can be formulated as:

$$\delta(x,t) = \frac{1}{\rho A} \sum_{j=1}^{\infty} \nu_j^{-2} \psi_j(x) \int_{-\infty}^{\infty} h_j(\tau) d\tau \int_0^l q(\xi, t-\tau) \psi_j(\xi) d\xi \tag{25}$$

$$q(x,t) = \sum_{j=1}^{\infty} q_j(t) \psi_j(x) \tag{26}$$

$$q_j(t) = \nu_j^{-2} \int_0^l q(x,t) \psi_j(x) dx, \quad \nu_j^2 = \int_0^l \psi_j^2(x) dx \tag{27}$$

where $h_j(t)$ is the impulse response function associated with the jth mode, $\psi_j(x)$ is the jth mode shape function:

$$\psi_j = \sin \frac{j\pi x}{\ell} \quad j=1,\infty \tag{28}$$

also

ρ = mass density per unit area

A = cross-sectional area = wt

Because there is no closed form solution, the distribution of $\delta(x,t)$ is extremely difficult to construct. But $\delta(x,t)$ is stationary; its mean and mean-square values (in time space only) are not a function of time. In this test case,

$$E[\delta(x,t)] = 0 \tag{29}$$

$$E[\delta^2(x,t)] = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \psi_j(x)\psi_k(x)\psi_j(a)\psi_k(a)\nu_j^{-2}\nu_k^{-2} I_{jk} \tag{30}$$

where

$$I_{jk} = \frac{S_0}{(\rho A)^2} \frac{\Phi(\omega_j,\omega_k;\omega_c) + \Phi(\omega_k,\omega_j;\omega_c)}{(\omega_j^2 - \omega_k^2)^2 + 2\beta^2(\omega_j^2 + \omega_k^2)} \tag{31}$$

$$\begin{aligned}\Phi(\omega_j,\omega_k;\omega_c) = {} & \frac{\omega_j^2 - \omega_k^2 - \beta^2}{2(\omega_j^2 - \beta^2/4)^{1/2}} \ln \frac{\omega_c^2 + \omega_j^2 - 2\omega_c(\omega_j^2 - \beta^2/4)^{1/2}}{\omega_c^2 + \omega_j^2 + 2\omega_c(\omega_j^2 - \beta^2/4)^{1/2}} \\ & + 2\beta\left\{\tan^{-1}\left[\frac{\omega_c - (\omega_j^2 - \beta^2/4)^{1/2}}{\beta/2}\right]\right. \\ & \left. + \tan^{-1}\left[\frac{\omega_c + (\omega_j^2 - \beta^2/4)^{1/2}}{\beta/2}\right]\right\}\end{aligned} \tag{32}$$

$$\beta = 2\zeta_j w_j = c/\rho A \tag{33}$$

where c is the damping coefficient and ζ_j is the damping factor of the jth mode.

The goal of the reliability analysis is to construct, at x=a, the distribution of the mean-square value defined in Eq. (30). (This distribution may be used to establish the distribution of δ, considering all random variables).

The cutoff frequency ω_c will be chosen such that

$$\omega_{14} < \omega_c$$

where ω_{14} is the mean value of the 14th natural frequency. Following is the list of the random variables and the deterministic parameters:

Random Variables: E, w, t, ℓ, c, ρ
Deterministic Value: a, S_0, ω_c

Based on the chosen mean or median values, the following nondimensional parameters will be chosen:

$$H = \frac{t}{\ell} = 0.01$$

$$\Omega_c = \omega_c \ell/(E/\rho)^{1/2} = 2\pi$$

$$B = \beta\ell/(E/\rho)^{1/2} = 0.02$$

$$\eta = \frac{a}{\ell} = 0.3$$

References

1. PSAM Proposal, Technical proposal, Part 2, Appendix C.11.

2. Leissa, A.W., Macbain, J.C. and Kielb, R.E., "Vibrations of Twisted Cantilevered Plates - Summary of Previous and Current Studies," submitted for publication in Journal of Sound and Vibration.

3. Elishakoff, I., Probabilistic Methods in the Theory of Structures, John Wiley, & Sons, Inc., 1983.

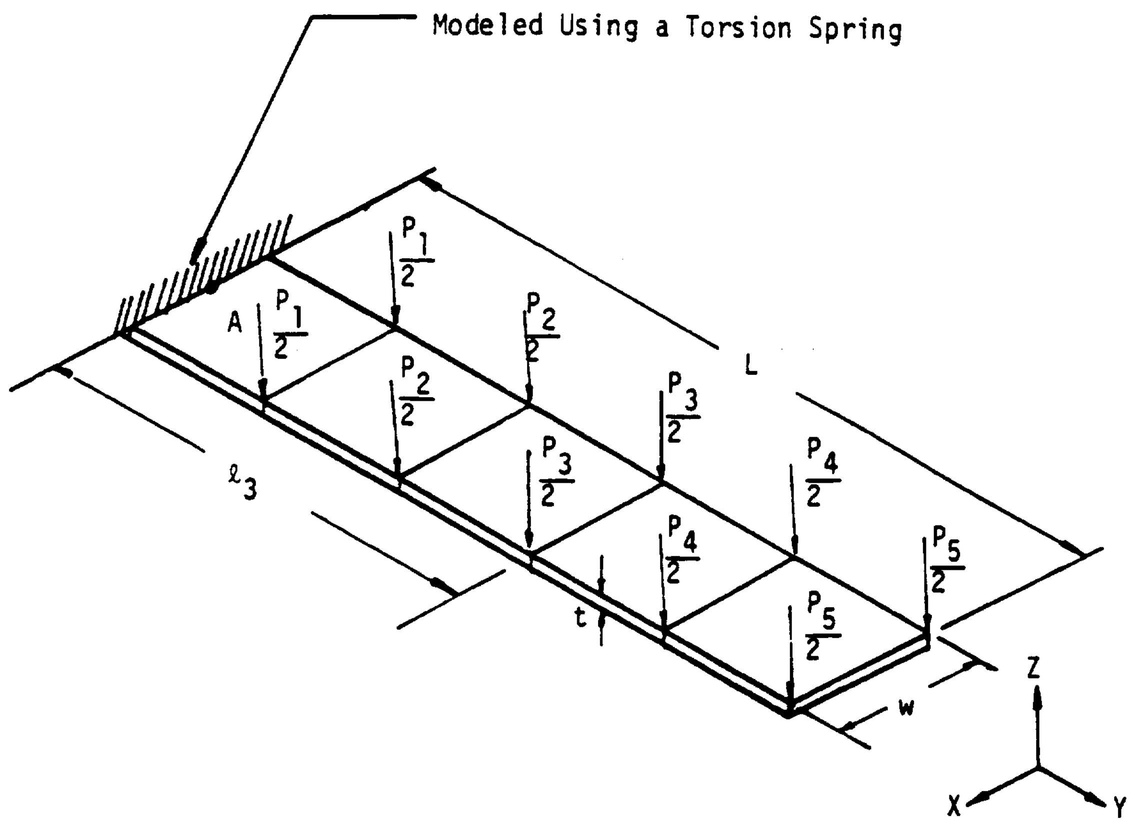

FIGURE 1. Model Definition of a Cantilever Beam

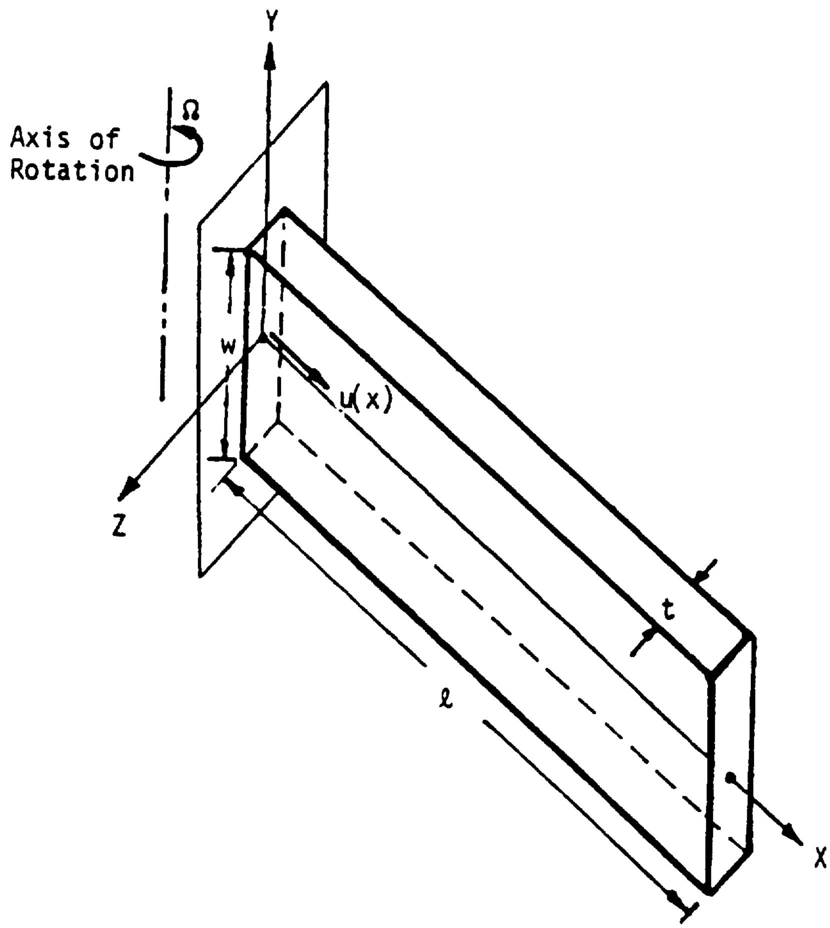

FIGURE 2. Rotating Beam Geometry

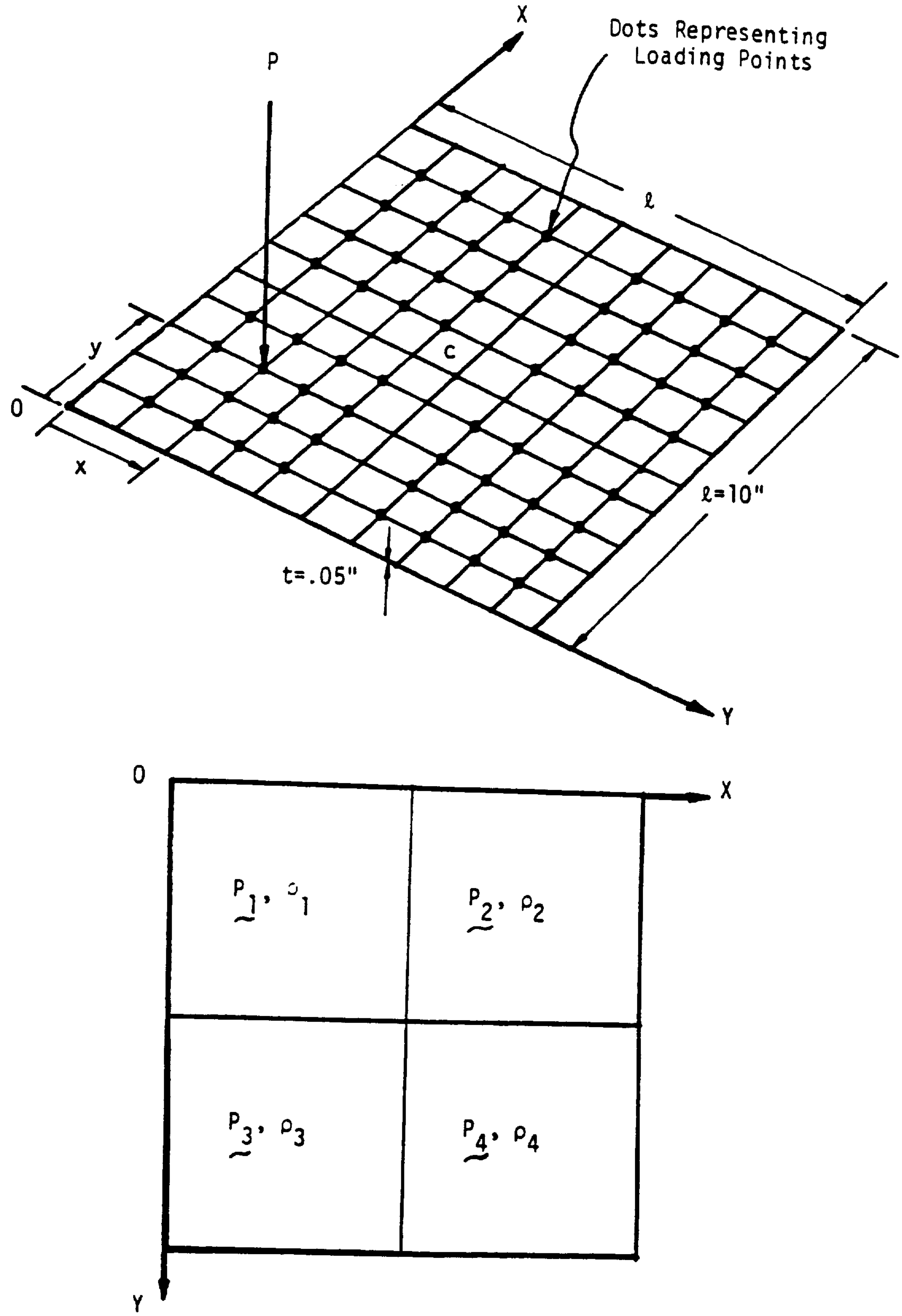

FIGURE 4. The Plate Model for Test Case 7

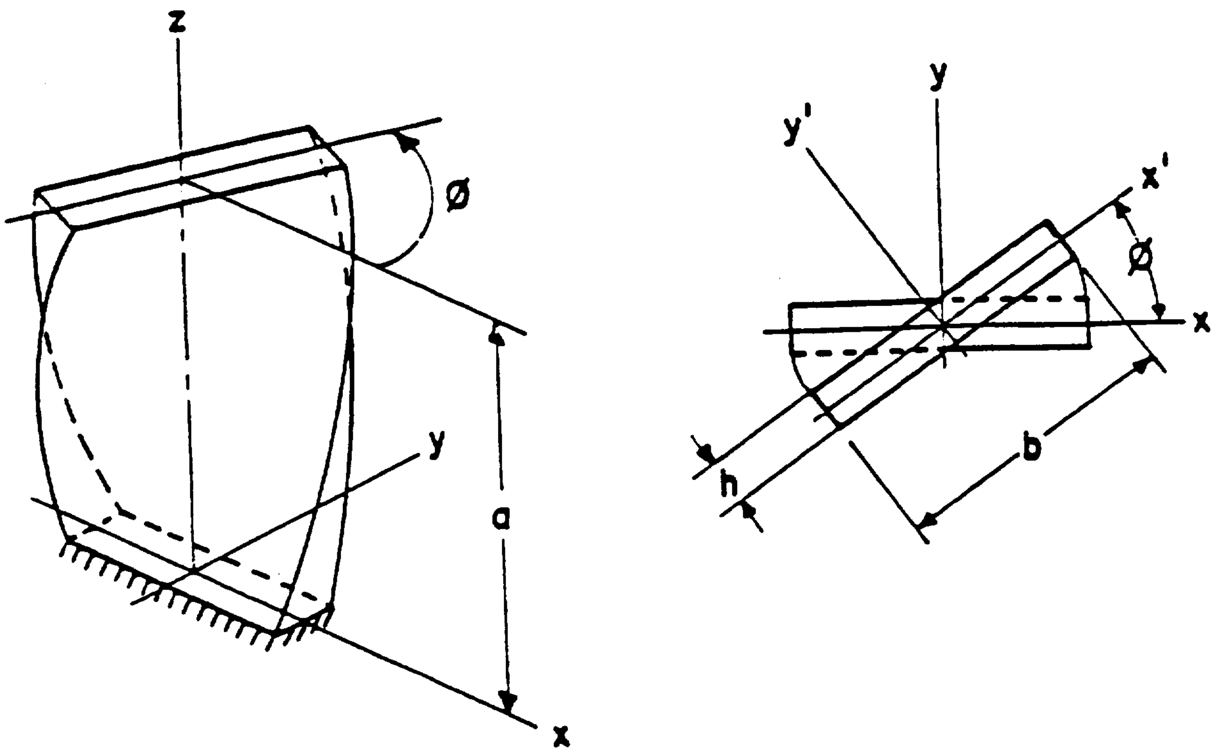

FIGURE 3. Twisted Cantilevered Plate

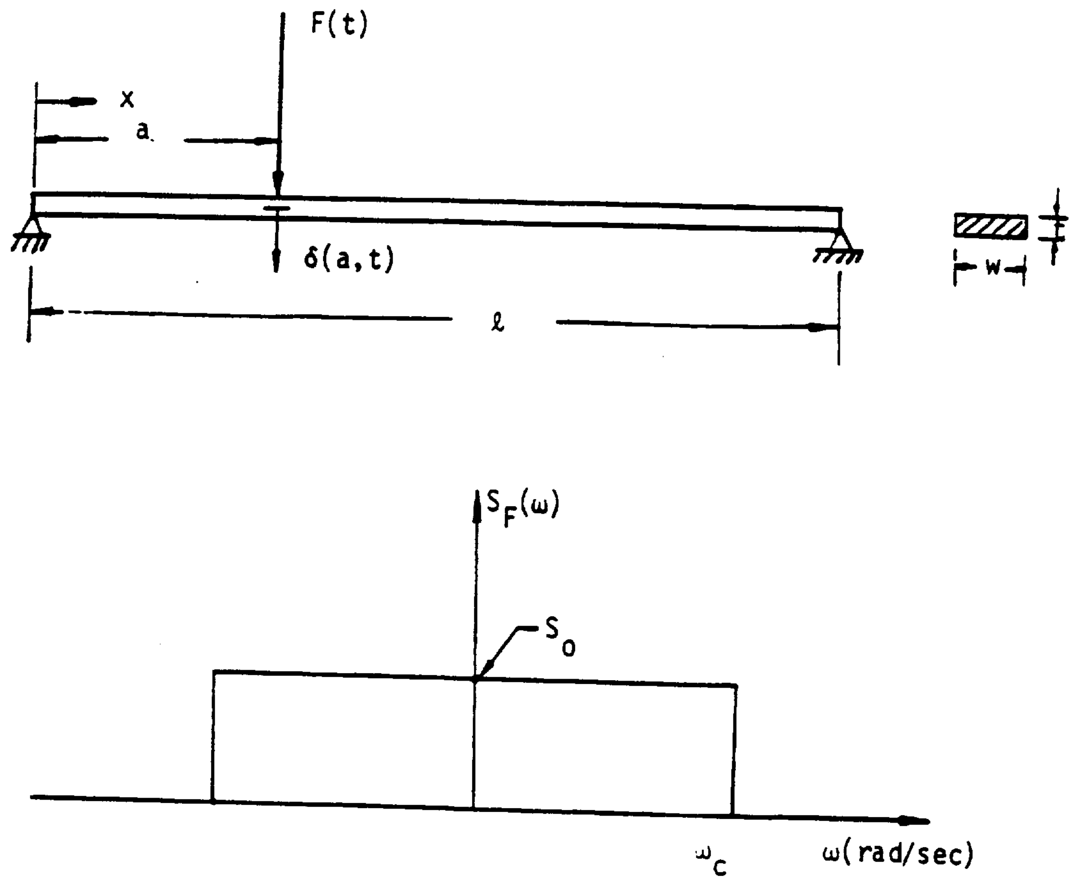

FIGURE 5. Simply Supported Beam Under Concentrated Random Force F(t)

Section 4 Appendix

Validation Case for the First Year PFEM Code

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November 1985

Introduction

A preliminary plan for validation of the first year PFEM code was described earlier in the PSAM Monthly Progress Report No. FY '85-12 (Attachment 4). Presented herein is the solution to the validation test Case 1.

Case 1

Problem: The cantilever beam modeled in Figure 1 is subjected to static loadings, $P_i (i=1,5)$. P_i's are correlated with the correlation coefficients defined as:

$$\rho_{P_i P_j} = \exp\left[-\left(\frac{\Delta X_{ij}}{L}\right)\right] \qquad (1)$$

where Δx_{ij} is the distance from element i to element j, and L is the length of the beam. The goal of the analysis is to determine:

1. The cdf (cumulative distribution function) and the pdf (probability density function) of the tip displacement, δ.
2. The cdf and the pdf of the maximum stress, S, at point A.
3. The probability that the stress S will exceed the yield strength, σ_y.

Solution: The covariance matrix of the correlated static loadings is

$$\underset{\sim}{C}_p = \sigma_{P_i}^2 \begin{bmatrix} 1 & e^{-.2} & e^{-.4} & e^{-.6} & e^{-.8} \\ & 1 & e^{-.2} & e^{-.4} & e^{-.6} \\ & & 1 & e^{-.2} & e^{-.4} \\ & & & 1 & e^{-.2} \\ & \text{SYMMETRIC} & & & 1 \end{bmatrix} \qquad (2)$$

where $\sigma_{P_i} = 2$ is the standard deviation of P_i.

The eigenvalues of $\underset{\sim}{C}_p$ are:

$$\underset{\sim}{\lambda} = \begin{Bmatrix} 14.98 \\ 0.4385 \\ 0.5941 \\ 2.973 \\ 1.059 \end{Bmatrix} \qquad (3)$$

and the normalized matrix of eigenvectors of $\underset{\sim}{C}_p$ is:

$$\underset{\sim}{A} = \begin{bmatrix} 0.4148 & -0.2083 & -0.3941 & -0.5871 & 0.5334 \\ 0.4621 & 0.5107 & 0.5871 & -0.3941 & -0.1599 \\ 0.4782 & -0.6256 & 0.0 & 0.0 & -0.6163 \\ 0.4621 & 0.5107 & -0.5871 & 0.3941 & -0.1599 \\ 0.4148 & -0.2083 & 0.3941 & 0.5871 & 0.5334 \end{bmatrix} \tag{4}$$

Using Eq. 4, an uncorrelated vector is generated as

$$\underset{\sim}{p} = \underset{\sim}{A}^T \underset{\sim}{P} \tag{5}$$

because P is a normal vector, p is also a normal vector. The mean values and the standard deviations of $\underset{\sim}{p}$ can be computed from Eq. 5 as follows.

$$\underset{\sim}{\mu}_p = \underset{\sim}{A}^T \underset{\sim}{\mu}_P = \begin{Bmatrix} 44.64 \\ -0.4158 \\ 0.0 \\ 0.0 \\ 2.6114 \end{Bmatrix} \tag{6}$$

$$\underset{\sim}{\sigma}_p = \sqrt{\underset{\sim}{\lambda}} = \begin{Bmatrix} 3.870 \\ 0.6622 \\ 0.7707 \\ 1.711 \\ 1.029 \end{Bmatrix} \tag{7}$$

In Figure 1, the tip displacement is

$$\delta = \sum_{i=1}^{5} \left[\frac{2P_i \ell_i^2}{Ewt^3}(3L - \ell_i) + \frac{P_i \ell_i L}{K}\right] \tag{8}$$

Noting that ℓ_i is related to L by

$$\ell_i = \frac{i}{5} L \tag{9}$$

Eq. 8 can be written as

$$\delta = \frac{L^3}{Ewt^3}\,[0.224P_1 + 0.832P_2 + 1.728P_3 + 2.816\,P_4 + 4P_5] + \frac{L^2}{K}\,[0.2P_1 + 0.4P_2 + 0.6P_3 + 0.8\,P_4 + P_5] \tag{10}$$

in which each P_i can be transformed to p_i using the inverse of Eq. 5. As an example, P_1 can be expressed as

$$P_i = 0.4148\,p_1 - 0.2083\,p_2 - 0.3941\,p_3 - 0.5871\,p_4 + 0.5334\,p_5 \tag{11}$$

By substituting Eq. 11, etc., into Eq. 10, the displacement becomes a function of p_1.

$$\delta = \frac{L^3}{EWt^3}\,[4.264\,p_1 - 0.09786\,p_2 + 0.3233\,p_3 + 2.999\,p_4 + 0.6045\,p_5] + \frac{L^2}{K}\,[1.339\,p_1 - 0.01248\,p_2 + 0.08044\,p_3 + 0.6273\,P_4 + 0.07842\,p_5] \tag{12}$$

where the ten random variables: L, E, w, t, K and p_i (i=1,5) are independent.

The second performance function considered is the maximum stress at the root section.

$$S = \frac{6 \sum_{i=1}^{5} P_i \ell_i}{wt^2} = \frac{L}{wt^2}\,[1.2P_1 + 2.4P_2 + 3.6\,P_3 + 4.8\,P_4 + 5.6\,P_5] \tag{13}$$

Using Eq. 5, Eq 13 becomes

$$S = \frac{L}{wt^2} [7.87\ p_1 + 0.00856\ p_2 + 0.3248\ p_3 + 3.529\ p_4 + 0.2567\ p_5] \tag{14}$$

where S is a function of eight independent random variables.

Note that further simplifications of Eq. 12 and Eq. 14 can be done. For example, Eq. 14 can be reduced to

$$S = UV \tag{15}$$

where $U = L/wt^2$ is a lognormal variable and V is a normal variable. However, reliability analyses using Eq. 12 and Eq. 14 provide more information about all the design variables involved (e.g., the "design point" provide the sensitivities of the variables). Therefore, Eq. 12 and Eq. 14 are considered better for checking purposes. it should also be mentioned that in the PFEM code, δ and S will be approximated by polynomial equations involving all the independent random variables. The accuracies of the approximating equations may be checked using Eq. 12 and Eq. 14.

The third performance function can be constructed as

$$g = \sigma_y - S \tag{16}$$

in which σ_y is the yield strength and S is the stress evaluated using Eq. 14. Thus, g is a function of nine independent random variables.

Using Eq. 12, Eq. 14 and Eq. 16, reliability analyses were performed using the FPI program as well as a Monte Carlo program. To check the Monte Carlo program, a sample size of 100,000 was used to evaluate the statistics of δ. The result is shown in Table 1 where the data of the ten random variables is listed. The median of the tip displacement is, from Table 1,

$$\delta(\text{Simulation}) = 0.40321 \text{ in.}$$

By substituting the medians of the random variables into Eq. 12, the displacement is computed as

$$\delta = 0.40319 \text{ in.}$$

which agrees with the simulation result.

The cdf of δ, computed at thirteen values of δ, is listed in Table 2, and is plotted on a normal probability paper (Figure 2). It shows that the results of the FPI analysis and the Monte Carlo simulation are close.

The pdf of δ, which is the derviative of the cdf of δ, must be evaluated numerically using the cdf values. Therefore, for validation purposes, it is more direct to compare the cdf's than to compare the pdf's. However, a pdf plot may be useful for making engineering judgements and decisions, therefore, the pdf should also be computed. For a convenient presentation of both the cdf and pdf, a method is suggested in the following for generating analytical expressions.

Employing the one-to-one mapping:

$$F(x) = \Phi(u) \tag{17}$$

where F(x) is the cdf of X (such as δ, S) and $\Phi(u)$ is the standard normal cdf in which u is the standardized normal variate. Assume that F(x) is known, u can be computed as

$$u = \Phi^{-1}[F(x)] \tag{18}$$

By taking the derivatives of Eq. (17), the pdf of X is

$$f(x) = \phi(u) \frac{du}{dx} \tag{19}$$

The next step is approximate u by a polynomial equation:

$$u = \sum_{i=0}^{n} a_i x^i \tag{20}$$

then the cdf and the pdf are approximated by

$$F(x) = \Phi[\sum_{i=0}^{n} a_i x^i] \tag{21}$$

and

$$f(x) = \phi[\sum_{i=0}^{n} a_i x^i] \cdot \sum_{i=1}^{n} i a_i x^{i-1} \tag{22}$$

where the approximating formula for $\Phi(\cdot)$ is available (e.g., in <u>Handbook of Mathematical Functions</u>, by Abramowitz and Stegun), and

$$\phi(u) = .3989 \exp[-0.5u^2] \tag{23}$$

The reason for establish Eq. 20, instead of generating

$$F(x) = \sum_{i=0}^{n} a_i x^i \tag{24}$$

is because the functional relationship between F(x) and x is, in general, difficult to approximate using Eq. 24. On the other hand, the relationship between u and x is usually not significantly nonlinear. For example, Figure 2 shows that u and δ are approximately linearly-related. Note that if X is normally distributed, then

$$u = \frac{X-\mu}{\sigma} = a_0 + a_1 x \tag{25}$$

in other words, u related to x linearly.

To establish analytical expressions of $F(\delta)$ and $f(\delta)$, a table is established in the following

δ	$u = \Phi^{-1}[F(\delta)]$
0.22	-3.649
0.26	-2.671
0.30	-1.818
0.34	-1.064
0.38	-0.391
0.42	0.213
0.46	0.764
0.50	1.270
0.54	1.736
0.58	2.164
0.62	2.564
0.66	2.940
0.70	3.290

where the absolute values of the u's are actually the safety indices of the FPI output. By using a curve-fitting program, the following fourth-degree polynomial is established:

$$u = \sum_{i=0}^{4} a_i \delta^i \tag{26}$$

where

a_0 = -12.1307

a_1 = -54.6602

a_2 = -90.7163

a_3 = 87.4329

a_4 = -34.9088

For the values of δ computed, the relative errors in u estimates are approximately less than one percent.

By substituting the coefficients of Eq. 26 into Eq. 22, it is now convenient to compute $f(\delta)$. Figure 3 shows the plot of $f(\delta)$ using Eq. 22.

The above procedure of presenting the result of the reliability analysis for the tip displacement can be applied for the maximum stress.

A Monte Carlo simulation with sample size of 100,000 resulted in

$$\tilde{S} = 7294 \text{ psi}$$

By substituting the medians of the random variables into Eq. 14, the stress is

$$S = 7330 \text{ psi}$$

which is near the simulation result.

Thirteen values of the cdf of S are computed and listed in Table 3. The result is also plotted on a normal probability paper (Figure 4). There is almost no difference between the simulation (sample size = 100,000) and the FPI results.

The analytical expressions of F(s) and f(s) are given by Eq. 17 and Eq. 19 where

$$u = \sum_{i=0}^{4} a_i S^i \qquad (27)$$

in which

a_0 = -12.1726

a_1 = 2.88859

a_2 = -0.245685

a_3 = 0.0127482

a_4 = -0.000278666

a f(s) plot using these coefficients is shown in Figure 5.

The probability that the stress exceeds the yield strength requires only one run of the FPI program. The result is

$$p_f = 0.0511 \text{ (FPI)}$$
$$= 0.0510 \text{ (Monte Carlo with sample size = 100,000)}$$

The pdf of the yield strength is also plotted in Figure 5 to compare with the pdf of the stress. The pdf of strength, which is Weibully-distributed, is

$$f(x) = (\frac{\alpha}{\beta})(\frac{x}{\beta})^{\alpha-1}\exp[-(\frac{x}{\beta})^{\alpha}]$$

where

$$\alpha = 12.0226$$
$$\beta = 10.4342 \text{ ksi}$$

Table 1. Statistics of Tip Displacement
for Test Case 1

MONTE CARLO SOLUTION

SAMPLE SIZE, K= 100000

NUMBER OF RANDOM VARIABLES, N= 10

RANDOM VARIABLES

VARIABLE	DISTRIBUTION	MEAN/*MEDIAN	STD DEV/*COV
E	LOGNORMAL	0. 10000E+08	0. 30000E-01
L	LOGNORMAL	0. 20000E+02	0. 50000E-01
t	LOGNORMAL	0. 98000E+00	0. 50000E-01
w	LOGNORMAL	0. 10000E+01	0. 50000E-01
K	LOGNORMAL	0. 10000E+06	0. 50000E-01
p1	NORMAL	0. 44643E+02	0. 38700E+01
p2	NORMAL	-0. 41580E+00	0. 66220E+00
p3	NORMAL	0. 00000E+00	0. 77075E+00
p4	NORMAL	0. 00000E+00	0. 17109E+01
p5	NORMAL	0. 26114E+01	0. 10290E+01

NOTE: MEDIAN AND COV FOR LOGNORMAL VARIABLES ONLY.

STATISTICS OF DISPLACEMENT:

MEAN = 0. 40884E+00

STD DEV = 0. 68587E-01

MEDIAN = 0. 40321E+00

COV = 0. 16776E+00

Table 2. The cdf of the Tip Displacement (Case 1)

Displacement inch	cdf	
	[a]Monte Carlo	FPI
0.22	0.000169	0.000132
0.26	0.00468	0.00378
0.30	0.0385	0.0346
0.34	0.152	0.144
0.38	0.362	0.348
0.42	0.596	0.585
0.46	0.785	0.777
0.50	0.901	0.898
0.54	0.9606	0.9586
0.58	0.9858	0.9848
0.62	0.9953	0.9948
0.66	0.99842	0.99836
0.70	0.99966	0.999498

a: Sample size = 100,000

Table 3. The cdf of the Maximum Stress (Case 1)

Stress ksi	cdf	
	[a]Monte Carlo	FPI
4.4	0.000590	0.000599
5.06	0.00890	0.00876
5.72	0.0545	0.0548
6.38	0.188	0.185
7.04	0.404	0.400
7.70	0.636	0.634
8.36	0.816	0.814
9.02	0.922	0.920
9.68	0.972	0.9703
10.34	0.99107	0.99023
11.00	0.99724	0.99712
11.66	0.99912	0.999219
12.32	0.99966	0.999803

a: Sample size = 100,000

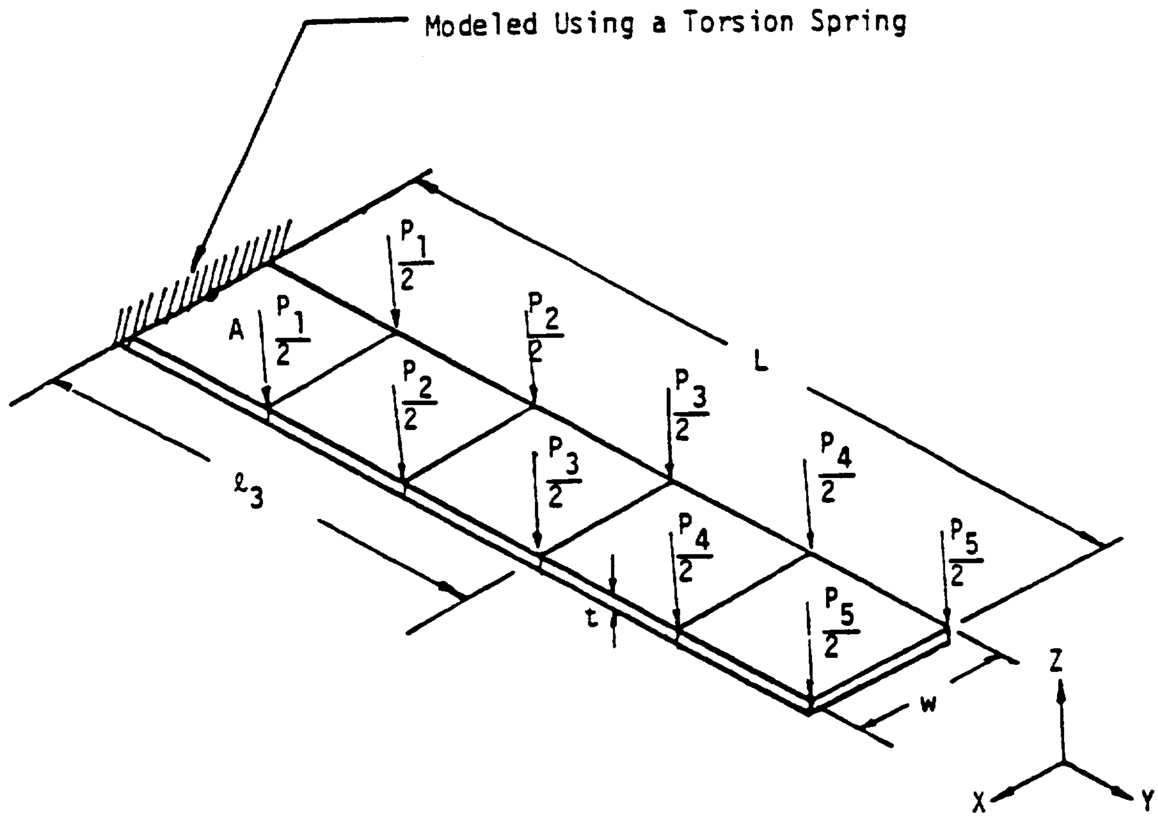
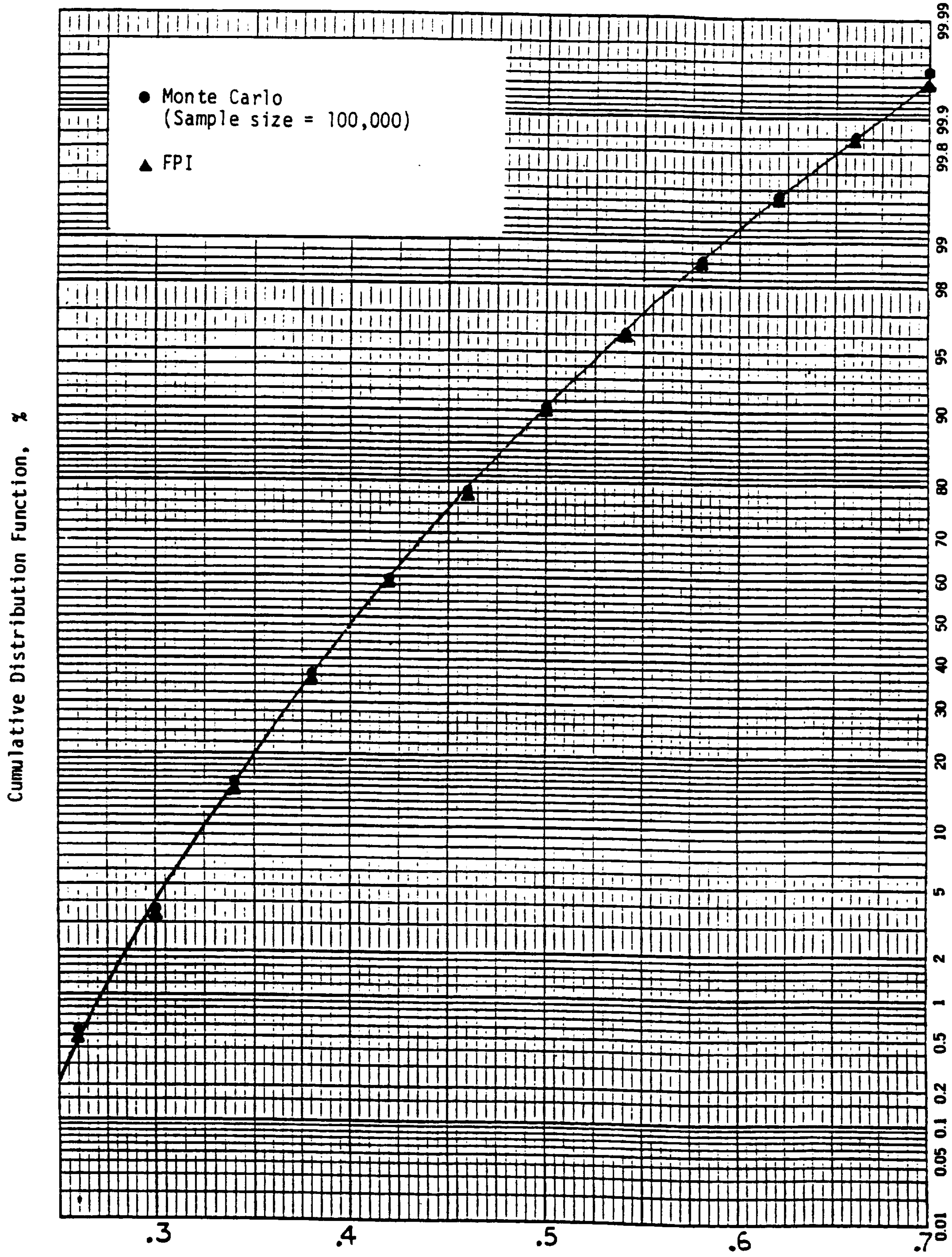


FIGURE 1. Model Definition of a Cantilever Beam

Figure 2. The cdf of the tip displacement (Case 1)



Displacement, inch

Figure 3. PDF of Tip Displacement (Case 1)

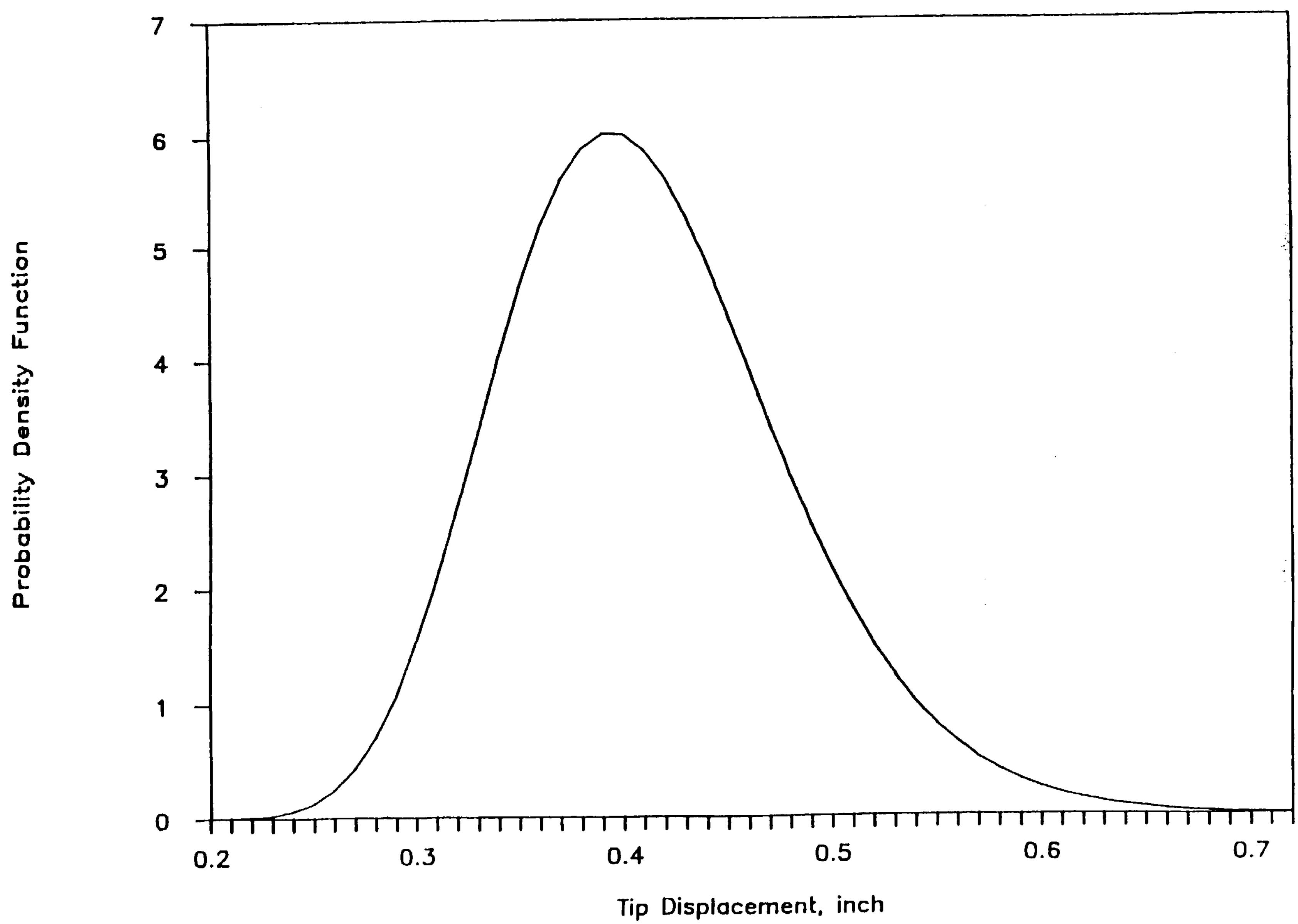


Figure 4. The cdf of the Maximum Stress (Case 1)

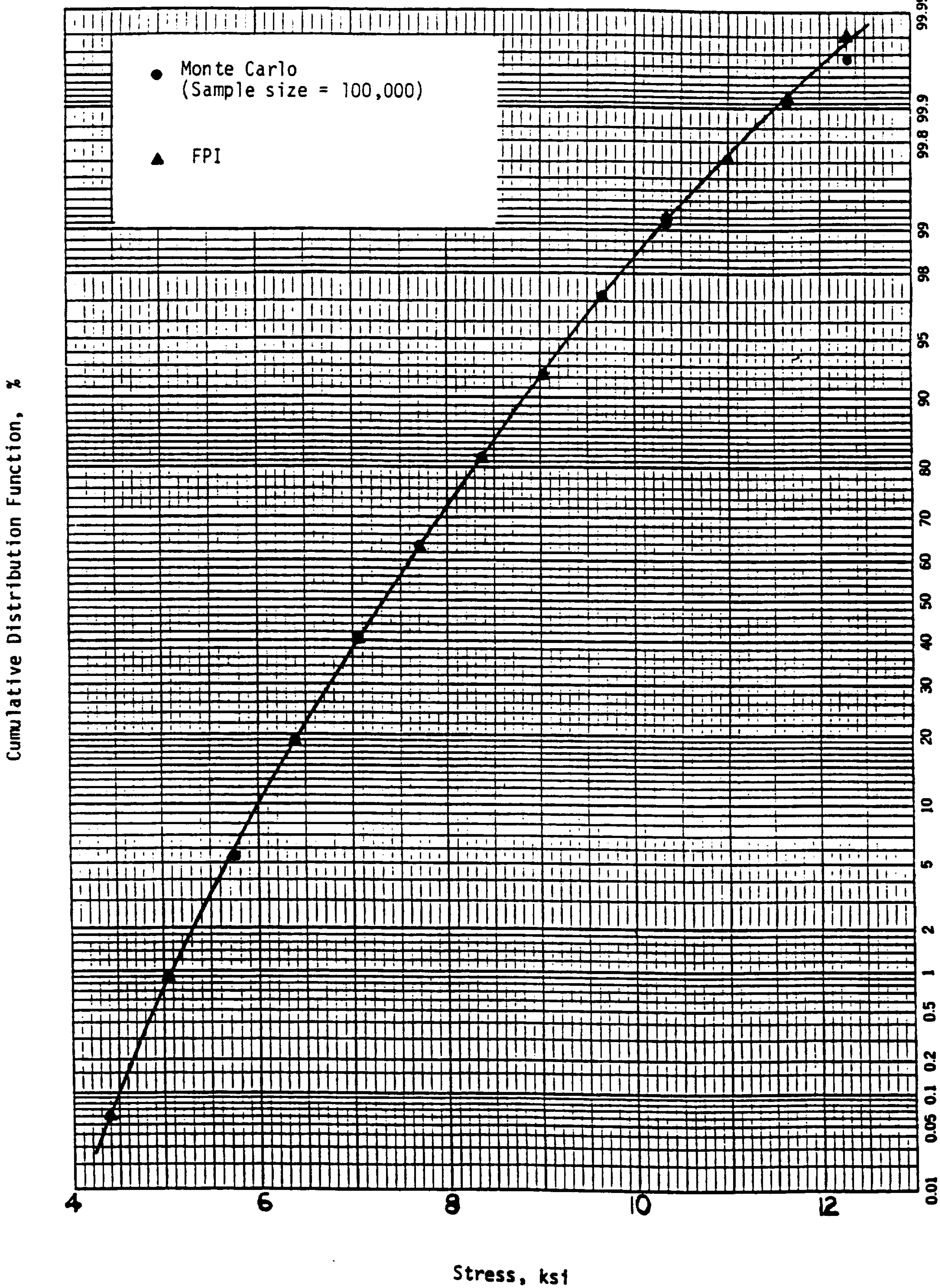
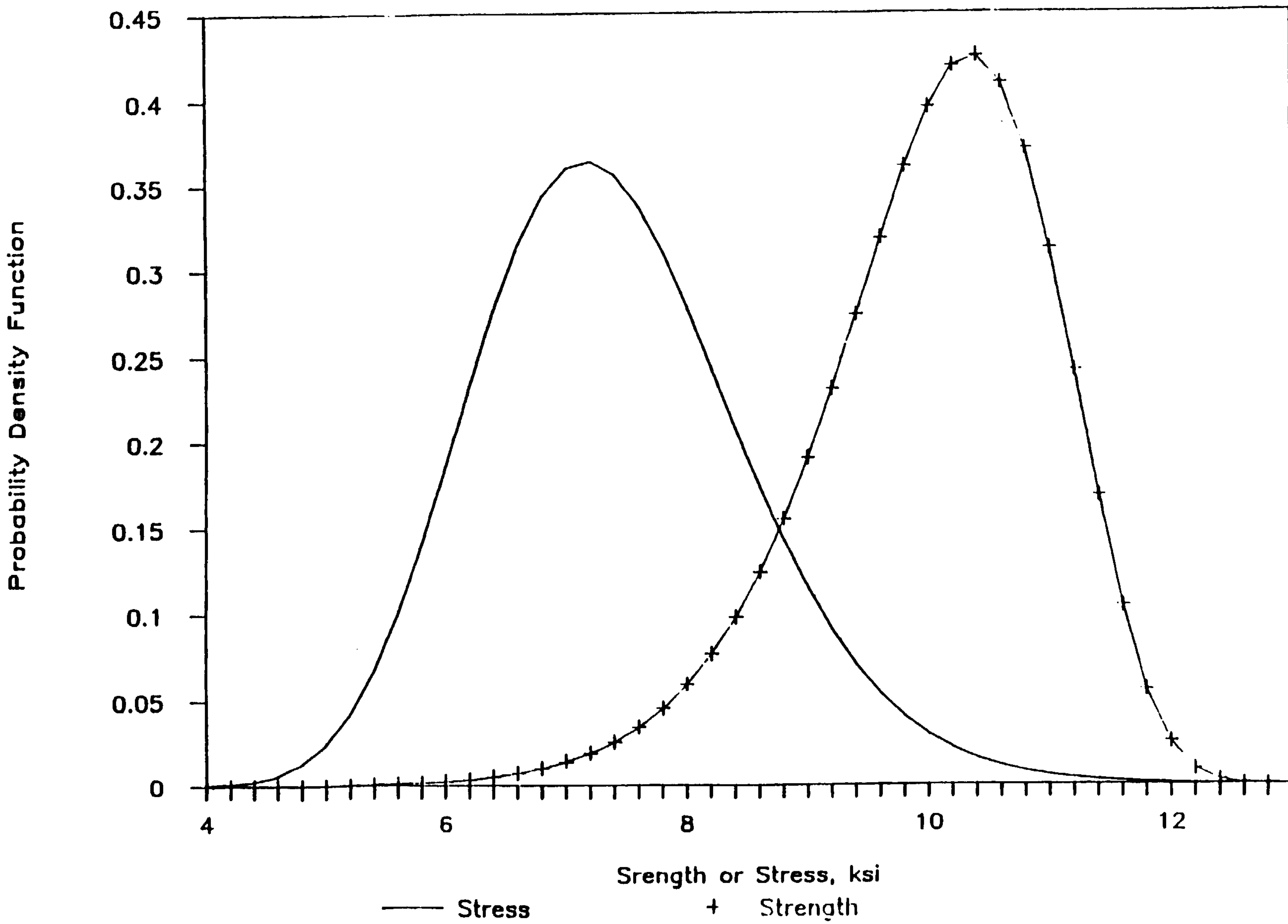
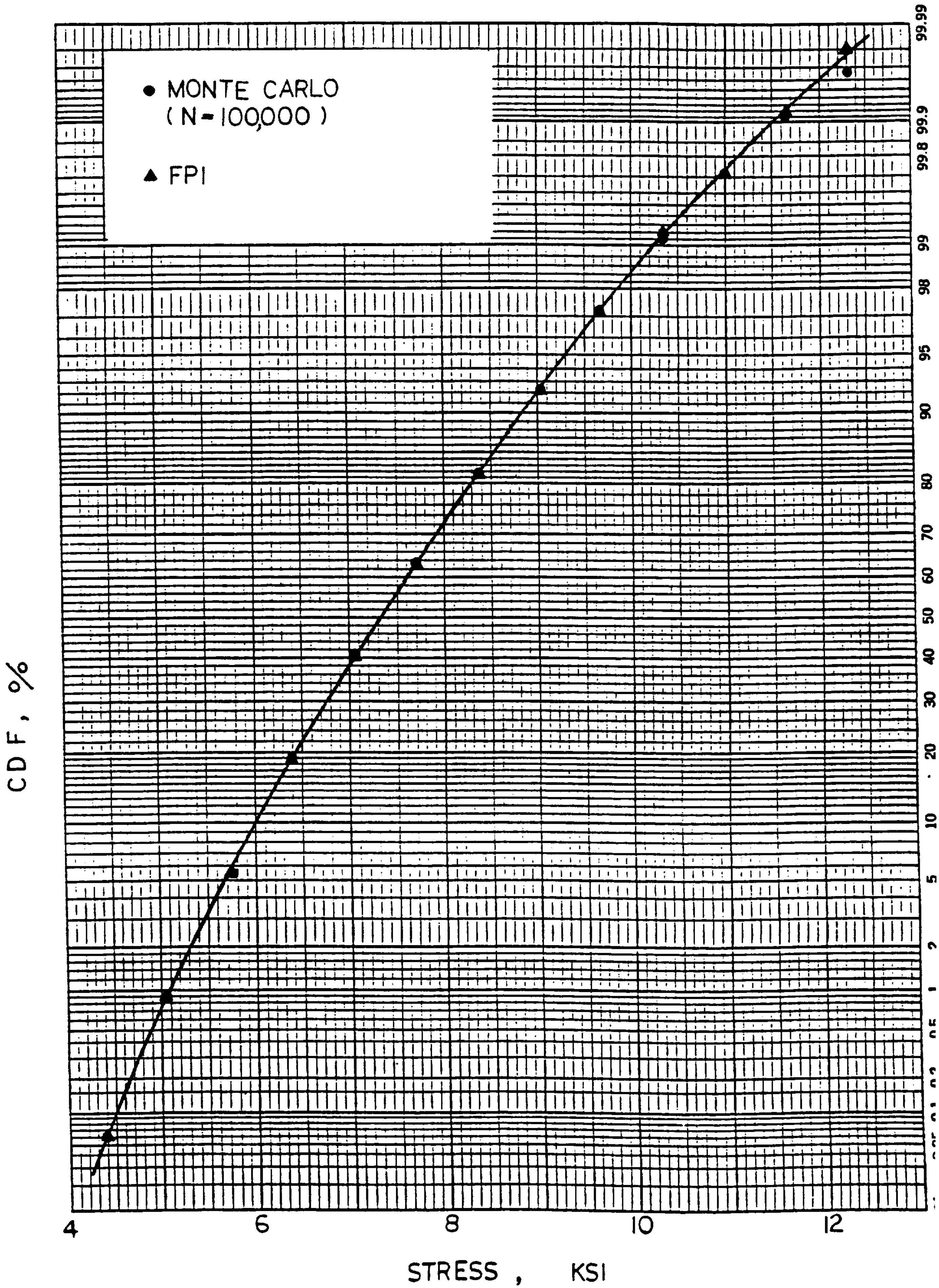


Figure 5. PDF's of Strength and Stress (Case 1)





MONTE CARLO
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FPI
CDF, %
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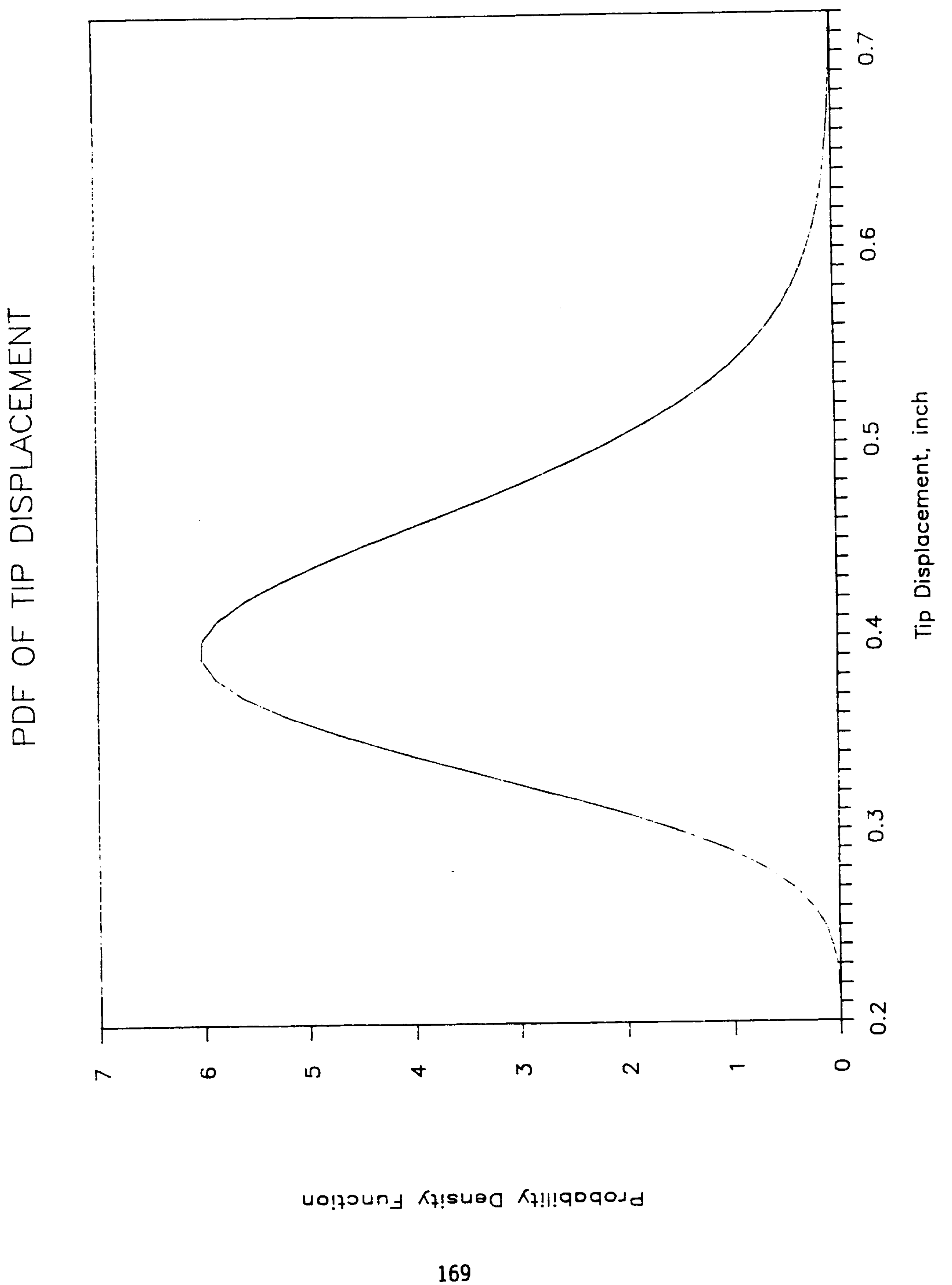
PDF OF TIP DISPLACEMENT
Probability Density Function
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Tip Displacement, inch
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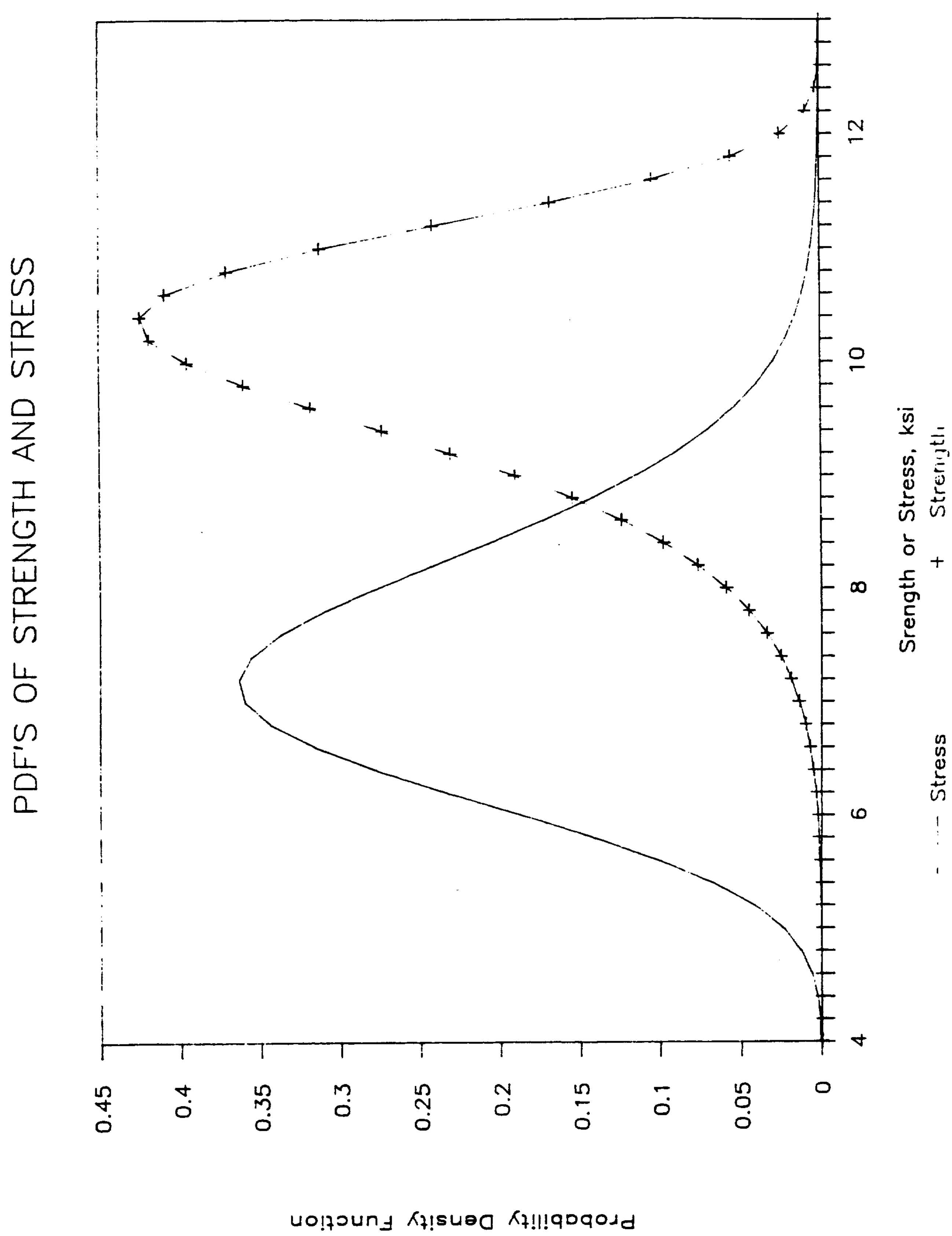
PDF'S OF STRENGTH AND STRESS
Probability Density Function
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0.4
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Srength or Stress, ksi
Stress

Section 5

Non-normal Correlated Vectors in Structural Reliability Analysis

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October 1985

Introduction

In structural reliability analysis, techniques are well developed for computing the probability measures of a given performance function, $g(X)$, where $\underset{\sim}{X}=(X_1,X_2, \ldots X_n)$ an <u>independent</u> vector. For example, fast probability integration (FPI) methods are currently being used in the PSAM project. However, it is not uncommon that the basic stochastic variables, X_i's, such as the geometrical parameters, the loadings, etc., are <u>dependent and non-normal</u>.

If the joint probability distribution function of the $\underset{\sim}{X}$ can be defined, the Rosenblatt transformation [1], as suggested by Hohenbichler and Rackwitz, may be employed to generate an independent, normal vector [2]. Then the FPI methods may be used. Unfortunately, in practical applications, the underlying joint distribution functions are very difficult to construct based on either theory or experiment. Moreover, the Rosenblatt transformation involves the inversion of the conditional distribution functions which are, in general, extremely difficult to derive or to compute numerically. This is particularly true for the cases involving large numbers of design random variables which are typical in the PSAM project.

Another way of solving the dependency problem is to use the marginal distributions and the correlation coefficients, which are relatively easy to obtain. It is well known that an orthogonal transformation can be employed to uncouple the dependency. If, in addition, the correlated variables are assumed normally distributed, then the transformed vector will be normal and independent. On the other hand, however, if the correlated variables cannot be assumed normally distributed, the distributions of the transformed independent vector are unknown and the FPI methods cannot be used.

The problem associated with non-normal correlated vector has been addressed in the literature [3,4]. For example, Der Kiureghian and Liu suggested that the bivariate distribution model due to Nataf [5] can be used. The transformed normal correlation coefficients between each two variables were found by iteratively solving a double integral equation. Because the calculation is tedious, a set of semi-empirical formulae for selected marginal distributions were developed.

In this study, a different approach employing series expansion is developed for solving the transformed normal correlation coefficients. The method is general and efficient, suitable for complicated distributions. Examples are provided to demonstrate the capabilities of the approximation method. Finally, the possible applications to the PSAM project are discussed.

Problem Definition

Given a non-normal vector $\underset{\sim}{X}$ with marginal distributions, i.e., the cumulative distribution functions (cdf's) $F_{X_i}(x_i)$ $(i=1, n)$, the covariance matrix $\underset{\sim}{C}_X$ may be constructed as

$$\underset{\sim}{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & & \\ C_{n1} & & \cdots & C_{nn} \end{bmatrix} \tag{1}$$

where

$$C_{ij} = E[X_i X_j] - E[X_i]E[X_j] \tag{2}$$

in which $E[\cdot]$ are the expected values. The correlation coefficients, $\rho_{X_i X_j}$, are defined as

$$\rho_{X_i X_j} = \frac{C_{ij}}{\sigma_i \sigma_j} \tag{3}$$

where σ_i and σ_j are the standard deviations (std.) of X_i and X_j, respectively.

Using the bivariate normal distribution model, the normal distributions are established first by the following transformations:

$$F_{X_i}(x_i) = \Phi_i(u_i) \qquad i=1,n \tag{4}$$

where $\Phi(\cdot)$ is the normal cdf and u_i is a standard normal variate. Note that Eq. 4 defines a one-to-one mapping; therefore, x_i may be formulated using the inverse transformation:

$$X_i = F_{X_i}^{-1}(\Phi_i(u_i)) \tag{5}$$

The inverse cdf's, i.e., $F_{X_i}^{-1}(\cdot)$ are available in closed forms for distributions such as Weibull and extreme value. For many distribution models, the closed form solutions do not exist; and for a given u value, x must be solved iteratively.

The next step is to find the correlation coefficients ρ_{ij} between any u_i and u_j ($i \neq j$). To simplify the presentation of the analysis, consider i=1 and j=2, and let $\rho = \rho_{12}$ (the "normal" correlation coefficient). ρ can be found by solving the following double integration equation.

$$\rho_{x_1 x_2} = \frac{1}{\sigma_1 \sigma_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1-\mu_1)(x_2-\mu_2)\phi_{12} du_1 du_2 \tag{6}$$

where μ_i = the mean values,

$$\phi_{12} = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1-\rho^2)}\right] \tag{7}$$

and X_1, X_2 are to be transformed to u_1, u_2 using Eq. 5. In general, there is no closed form solution for Eq. 6, and the calculation of ρ requires iteratively solving Eq. 6. The process is particularly cumbersome when the inverse transformation (i.e., Eq. 5) also needs to be solved iteratively.

Nevertheless, if all the ρ's are obtained for the corresponding $\rho_{X_i X_j}$, which means that the covariance matrix of $\underline{u}$ vector is established, then an orthogonal transformation may be employed to construct an independent normal vector suitable for reliability analysis. In the following discussion, an alternative procedure for computing ρ's will be developed which avoids the use of the double integral.

<u>Obtaining the Normal Correlation Coefficients Using Series Expansion - A New Approach</u>

Consider two correlated random variables, denoted as X_1 and X_2. The correlation coefficients $\rho_{X_1 X_2}$ can be computed as

$$\rho_{X_1 X_2} = \frac{E[X_1 X_2] - E[X_1]E[X_2]}{\sigma_1 \sigma_2} \tag{8}$$

Define the transformation from X_i to u_i as

$$X_i = T_i(u_i) \qquad i=1,\ 2 \tag{9}$$

and define

$$H(u_1,\ u_2) = X_1 X_2 = T_1(u_1) T_2(u_2) \tag{10}$$

Eq. 8 may be expressed as

$$\rho_{X_1 X_2} \sigma_1 \sigma_2 = E[H] - E[T_1]E[T_2] \tag{11}$$

Expand H into Taylor series about the point $(u_1, u_2) = (0,0)$,

$$H = H_{00} + \frac{1}{2}(H_{20}u_1^2 + 2H_{11}u_1u_2 + H_{02}u_2^2) + \text{H.O.T.} \tag{12}$$

where H.O.T. = Higher Order Terms

$$H_{ij} = \left.\frac{\partial^{i+j}H}{\partial u_1^i \partial u_2^j}\right|_{\underset{\sim}{u}=0} = \left.\frac{d^iT_1}{du_1^i} \cdot \frac{d^jT_2}{du_2^j}\right|_{\underset{\sim}{u}=0} \tag{13}$$

Eq. 11 can be written as

$$\begin{aligned}\rho_{X_1X_2}\sigma_1\sigma_2 = {} & H_{00} + \frac{1}{2} H_{20}E[u_1^2] + 2H_{11}E[u_1u_2] + H_{02}E(u_2^2) + E[\text{H.O.T}] \\ & - E[T_1] - E[T_2]\end{aligned} \tag{14}$$

The expected values of $u_1^iu_2^j$ can be derived using the moment generating function. For example, it can be shown that:

$$\begin{aligned} E[u_1u_2] &= \rho \\ E[u_1^2] &= 1 \\ E[u_1^3] &= 0 \\ E[u_1^2u_2^2] &= 1 + 2\rho^2 \\ &\vdots \\ &\text{etc.} \end{aligned} \tag{15}$$

A derivation of $E[u_1^iu_2^j]$ up to $i+j=8$ has been carried out and is listed in Table 1. By using Eq. 15, Eq. 14 simplifies to

$$\begin{aligned}\rho_{X_1X_2}\sigma_1\sigma_2 = {} & H_{00} + \frac{1}{2}(H_{20} + 2\rho H_{11} + H_{02}) \\ & + E[\text{H.O.T.}] - E[T_1] - E[T_2]\end{aligned} \tag{16}$$

Applying the condition that $\rho=0$ when $\rho_{X_1X_2}=0$, Eq. 16 further reduces to

$$\rho_{X_1X_2}\sigma_1\sigma_2 = \rho H_{11} + E[H.O.T.] \tag{17}$$

Therefore, a rough approximation, neglecting the fourth and higher-order (third-order coefficient is zero) terms, is simply

$$\rho = \frac{\sigma_1\sigma_2}{H_{11}} \rho_{X_1X_2} \tag{18}$$

in which

$$H_{11} = \frac{dT_1}{du_1} \cdot \frac{dT_2}{du_2} \bigg|_{\underset{\sim}{u}=0} \tag{19}$$

where T_i defines the transformations, e.g., Eq. 5.

If T_i are linear, e.g., when X_i are normal distributions, with means μ_i and standard deviations σ_i, then

$$X_i = T_i(u) = \mu_i + u_i\ \sigma_i \tag{20}$$

Employing Eq. 19 gives

$$H_{11} = \sigma_1\sigma_2 \tag{21}$$

and Eq. 8 then degenerates to

$$\rho = \rho_{X_1X_2} \tag{22}$$

as expected.

In general, T_i are non-linear functions of u_i, therefore, Eq. 18 is a good approximation only when X_i are not significantly non-normal. Using Table 1, a more complete approximation formula, up to eighth-order terms of H series was derived as:

$$\rho_{X_1X_2}\sigma_1\sigma_2 = \rho[H_{11}+ \frac{1}{2}(H_{13}+H_{31})+ \frac{1}{8}(H_{15}+2H_{33}+H_{51}) + \frac{1}{48}(H_{17}+3H_{35}+3H_{53}+H_{71})]$$
$$+ \frac{\rho^2}{2}[H_{22} + \frac{1}{2}(H_{24}+H_{42}) + \frac{1}{8}(H_{26}+2H_{44}+H_{62})]$$
$$+ \frac{\rho^3}{6}[H_{33} + \frac{1}{2}(H_{53}+H_{35})] \qquad (23)$$
$$+ \frac{\rho^4}{24}[H_{44}]$$

This equation is believed to be adequate for the practical problems involving highly non-linear transformations. The inclusions of even higher-order terms are straight forward using Eq. 23.

The procedure of computing ρ may be outlined in the following steps, suitable for computer programming:

1. Select N (say N=9) points of u values, e.g., from -4 to +4, with increments of 1. Compute $X_i=T(u_i)$ (i=1, 2) for the N values of u.

2. For both T_1 and T_2, find the (N-1)th order approximating polynomials using proper numerical schemes.

3. Compute H_{ij} using the result of step 2.

4. Solve ρ from Eq. 23.

For small coefficient of variations (COV's), say COV < 0.15, typical of the engineering problems, the fourth-order approximation provides a relatively efficient way of computing ρ. The equation is

$$\rho_{X_1X_2}\sigma_1\sigma_2 = \rho[H_{11}+\frac{1}{2}(H_{31}+H_{13})] + \frac{\rho^2}{2} H_{22} \qquad (24)$$

The highest derivatives for X_1 and X_2 are the third orders and ρ may be found by solving a quadratic equation.

Examples

To demonstrate how to use the derived formula, and to examine the accuracy, consider the following examples.

Example 1 - X_1 and X_2 are both lognormally distributed, with equal COV's = C. The transformations are:

$$F_{X_i}(x_i) = \Phi\left(\frac{\ln X_i - \mu_{Y_i}}{\sigma_{Y_i}}\right) = \Phi(u_i) \tag{25}$$

or

$$X_i = \exp[\mu_{Y_i} + u_i \sigma_{Y_i}] \tag{26}$$

where $Y_i = \ln X_i (i=1,2)$ and

$$\mu_{Y_i} = \ln \tilde{X}_i = \ln \frac{\mu_i}{\sqrt{1+C^2}} , \tag{27}$$

$$\sigma_{Y_i} = \sqrt{\ln(1+C^2)} = \sigma_Y \tag{28}$$

where $\tilde{X}_i$ is the median of X_i.

From Eq. 26

$$\frac{d^n x_i}{du_i^n} = \sigma_{Y_i}^n \exp[\mu_{Y_i} + u_i \sigma_Y] \tag{29}$$

It follows that

$$H_{ij} = (\sigma_Y)^{i+j} \exp[\mu_{Y_1} + \mu_{Y_2}] \tag{30}$$

Substituting Eq. 30 into Eq. 23 without truncating the series, and using Eq. 27, it can be shown that

$$\rho_{X_1 X_2} C^2 = \exp[\rho \sigma_Y^2] - 1 \tag{31}$$

Therefore,

$$\rho = \frac{\ln(1+\rho_{X_1X_2}C^2)}{\ln(1+C^2)} \tag{32}$$

which is an exact solution. The more general solution for the case $C_1 \neq C_2$ derived using a similar approach is

$$\rho = \frac{\ln(1+\rho_{X_1X_2}C_1C_2)}{\sqrt{\ln(1+C_1^2)\ln(1+C_2^2)}} \tag{33}$$

In general, closed-form solutions may be very difficult to derive or simply doesn't exist. In this case, a proper truncating series may be used. In the following, the effect of nonlinear transformation and the effect of truncating the series will be discussed using the two lognormal variables case.

Recall that the nonlinear transformation of a lognormal variable X to a normal u is

$$X = \tilde{X}\exp[u\ln\sqrt{1+C^2}\,] \tag{34}$$

Ignoring the constant $\tilde{X}$, the functional relationship between X and u is plotted in Figure 1 for three values of C, namely C=0.1, C=0.3 and C=0.5. It shows clearly that the C values significantly affect the nonlinearity of X=T(u) around u=0.

Now consider the truncated series. Assume that $C_1 = 0.1$ and $C_2 = 0.5$, the exact solution is:

$$\rho = 21.222\ \ln[1+.05\rho_{X_1X_2}] \tag{35}$$

and the approximating solution, using Eq. 23, is

$$.05618\rho_{X_1X_2} = \rho[.04712+.005492+.00032+.00001243]$$

$$+ \rho^2[.00111+.0001294+.00000754]$$

$$(36) \qquad + \rho^3[.00001744+.00000203]$$

$$+ \rho^4[.000000205]$$

where it is evident that the series converges quickly. As an example, let $\rho_{X_1X_2} = 0.9$, the results are as follows:

ρ (Second-order approx.) = 1.192

ρ (Fourth-order approx.) = 0.9423

ρ (Sixth-order approx.) = 0.9345

ρ (Eighth-order approx.) = 0.9341

ρ (Exact) = 0.9341

It may be observed that (for $\rho_{X_1X_2} > 0$ case)

$$\frac{\sigma_1\sigma_2}{H_{11}}\rho_{X_1X_2} > \rho_{exact} > \rho_{X_1X_2} \tag{37}$$

which means that the second-order approximation (using H_{11} and σ_i only) provides an upper bound of exact ρ. Note that $|\rho|(\text{Exact}) \geq |\rho_{X_1X_2}|$ has been proven (e.g., see Lancaster [6]). An important application of Eq. 37 is that if the bounds are judged narrow enough, then it is not necessary to try to obtain very accurate ρ value.

<u>Example 2</u>. Consider the case where one (say X_1) of the two variables is normally distributed, then

$$\frac{dX_1}{du_1} = \sigma_1 \tag{38}$$

$$\frac{d^n x_1}{du_1^n} = 0 \text{ for } n > 1 \tag{39}$$

at $u_1=0$.

The approximating series of Eq. 23 simplifies to

$$\begin{aligned} \rho_{X_1X_2}\sigma_2 &= \frac{\rho}{\sigma_1}[H_{11}+\frac{1}{2}H_{13}+\frac{1}{8}H_{15}+\frac{1}{48}H_{17}] \\ &= \rho[\frac{dx_2}{du_2}+\frac{1}{2}\frac{d^3x_2}{du_2^3}+\ldots]\Bigg|_{u_2=0} \end{aligned} \tag{40}$$

where ρ is not a function of X_1.

Assume that X_2 is a lognormal variable with median of unity and COV of 0.5. Pretend that the differentiation of x, with respect to u, is difficult and therefore must be done numerically. Using the strategy suggested earlier, nine sets of solutions are obtained as follows:

Set	u_2	$T(u_2)$
1	-4	0.1511
2	-3	0.2424
3	-2	0.3887
4	-1	0.6235
5	0	1.0
6	1	1.6038
7	2	2.5722
8	3	4.1253
9	4	6.6162

The function $T(u_2)$ was plotted in Figure 1.

The next step is to construct an eighth-order polynomial denoted as

$$x_2 = \sum_{n=0}^{8} A_n u_2^n \tag{41}$$

The required derivatives for computing ρ are

$$\left.\frac{d^n x_2}{du_2^n}\right|_{u_2=0} = A_n \cdot n! \tag{42}$$

where n=1, 3, 5 and 7.

By solving nine simultaneous linear equations, the coefficients are found as:

$$A_1 = 0.472353$$

$$A_3 = 0.017595$$

$$A_5 = 0.000191$$

$$A_7 = 0.000013$$

Using Eq. 42 and Eq. 40, the approximation solution is

$$\rho = 1.0584\ \rho_{X_1X_2} \tag{43}$$

The exact solution can be derived as:

$$\rho = \frac{C_2}{\sqrt{\ell n(1+C_2^2)}}\ \rho_{X_1X_2} = 1.0584\ \rho_{X_1X_2} \tag{44}$$

which proves that the proposed algorithm works very well.

Applications

Before discussing the possible applications, it is worthwhile to note that: (a) the correlations between the design variables may have significant effect on structural analysis (e.g., see Thoft-Christensen [6]), and (b) in probabilistic finite element analysis, the loading as well as the geometry must be discretized. For small element size, correlations usually exist between adjacent elements.

Let us consider a long bar. The cross-sectional area may be treated as a random variable. By discretizing the bar into n elements, there are n areas, A_i, each of which is a random variable. If the element lengths are relatively short, then it would be unrealistic to assume that the adjacent A_i are independent because the independency suggests the sudden changes in areas. On the other hand, A_i's cannot be assumed perfectly correlated if, in reality, A_i's are changing along the bar.

A possible solution to this problem is to treat the area (along the bar) as a random process. If the bars are manufactured under quality control, then it seems reasonable to further assume that the random process is stationary. Under the above assumptions, the marginal distribution as well as the correlations may be extracted from the measurement data. Obviously, the area need not be normally distributed.

The above discussion may be extended to two dimensional problems. For example, the thickness of a nominally flat plate may be treated as a stationary random process. The correlation functions may be constructed along different directions. This approach is, in fact, very similar as in defining a correlation field of random loading.

The treatment of the material property, e.g., modulus of elasticity (again, may be non-normal) as a correlated; but not perfectly correlated, random field is much more involved because it would be difficult to obtain experimental data for small elements. Correlation function needs to be assumed or established using other material properties which are related to the strength of material (e.g., Brinell Hardness Number). For the PSAM project, the selections of the correlations need to be tailored to the specific problem under investigation. For example, the modulus of elasticity of a turbine blade may be considered perfectly correlated. However, some independency may be assumed among different blades.

Assuming that the correlation functions are defined for the elements, the correlated non-normal variables can be transformed to independent normal variables using the procedure proposed earlier. Because the number of variables may be large, it is suggested that the orthogonal transformation should be done on a zone-by-zone basis where a zone is defined as a region in which the variables are correlated; there is negligible correlation outside this region. Since the number of variables in each zone is relatively small, the computation time may be reduced significantly.

Summary

Given a non-normal dependent vector with marginal distributions and correlation coefficients, a method using series expansion was developed for obtaining approximating correlation coefficients of the transformed joint normal distributions. Then the orthogonal transformation may be implemented to obtain independent normal vectors suitable for FPI or other reliability analysis.

Several levels of approximations were obtained and their accuracies and usefulness discussed. For example, the second-order approximation (using only H_{11}), which is easy to obtain, may provide a close bound of the exact solution. The series expansion has been derived up to eighth-order, which should be adequate for the problems encountered in the PSAM project. A simple numerical algorithm was suggested for computer programming.

Finally, in discussing the applications of the developed method, it was suggested that the geometries, the material properties, etc., may be treated as non-normal dependent vectors, and that the orthogonal transformation should be performed on a zone-by-zone basis.

References

1. Rosenblatt, M., "Remarks on a Multivariate Transformation," Annals of Mathematical Statistics, Vol. 23, No. 3, Sept. 1952, pp. 470-472.

2. Hohenbichler, M., and Rackwitz, R., "Non-Normal Dependent Vectors in Structural Safety," Journal of the Engineering Mechanics Division, ASCE, Vol. 100, No. EM6, Dec. 1981, pp. 1227-1238.

3. Der Kiureghian, A., and Liu, P.-L., "Structural Reliability Under Incomplete Probability Information," Report No. UCB/SESM-85/01, Department of Civil Engineering, University of California, Berkeley.

4. Grigoriu, M., "Approximate Analysis of Complex Reliability Problems," Structural Safety, 1, 1983, pp. 277-288.

5. Johnson, N.L., and Kotz, S., "Distributions in Statistics - Continuous Multivariate Distributions," John Wiley & Sons, Inc., New York, N.Y., 1976.

6. Lancaster, H.O., "Some Properties of Bivariate Normal Distribution Considered in the Form of a Contingency Table," Biometrika, Vol. 44, pp. 289-292, June 1957.

7. Thoft-Christensen, P., and Baker, M. J., "Structural Reliability Theory and Application," Springer-Verlag, 1982.

Table 1

Expected Values of the Functions of Two Correlated Standard Normal Variables (u, v)

FUNCTION	ORDER	EXPECTED VALUE
u, v	2	0
u^2	2	1
uv	2	ρ
u^3, v^3, u^3v, uv^2	3	0
u^4, v^4	4	3
u^3v, uv^3	4	3ρ
u^2v^2	4	$1+2\rho^2$
u^5, v^5, u^4v, uv^4, etc.	5	0
u^6, v^6	6	15
u^5v, uv^5	6	15ρ
u^4v^2, u^2v^4	6	$3+12\rho^2$
u^3v^3	6	$9\rho+6\rho^3$
u^7, v^7, u^6v, uv^6, etc.	7	0
u^8, v^8	8	105
u^7v, uv^7	8	105ρ
u^6v^2, u^2v^6	8	$15+90\rho^2$
u^5v^3, u^3v^5	8	$45\rho+60\rho^3$
u^4v^4	8	$9+72\rho^2+24\rho^4$

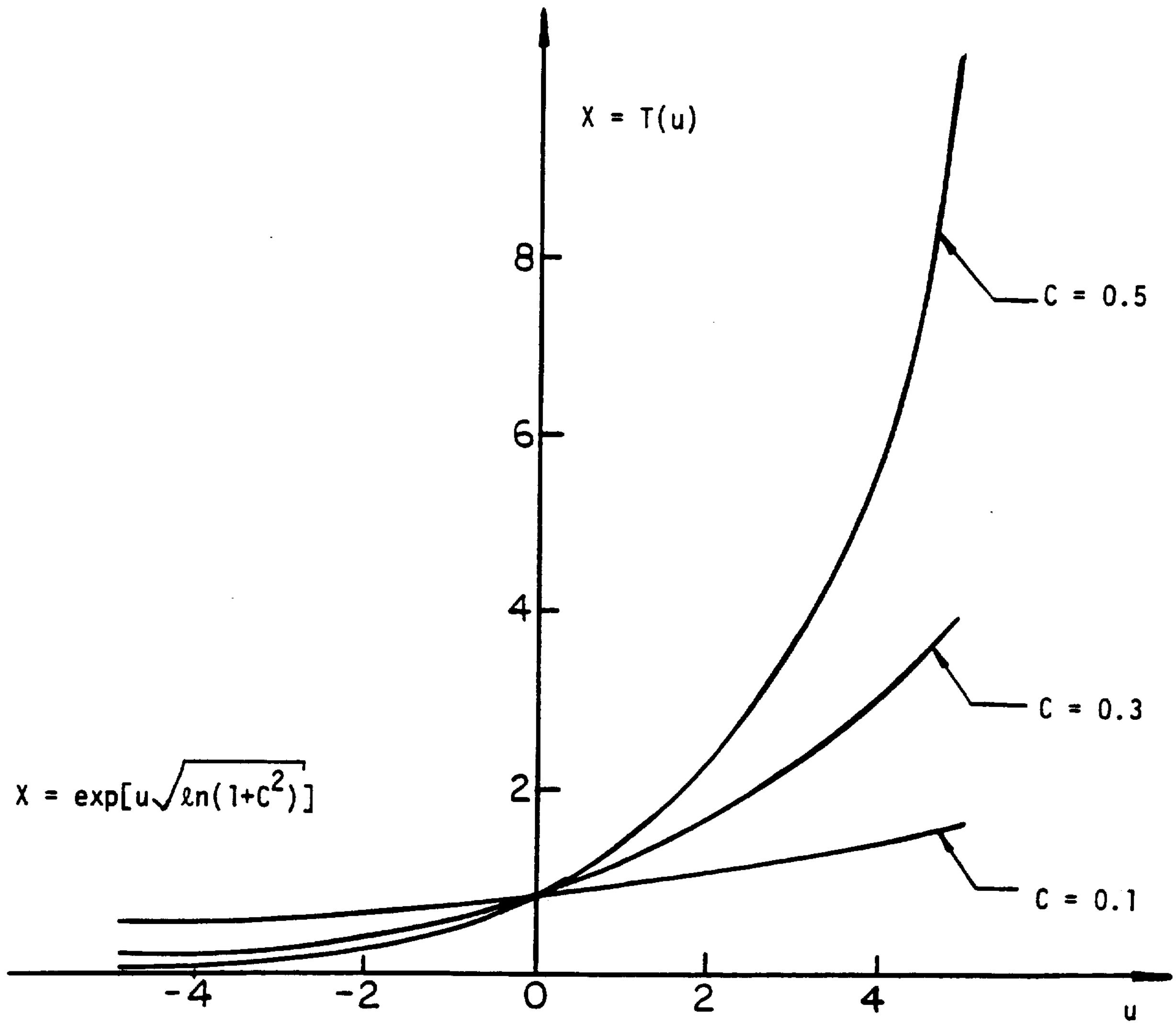

Figure 1. Nonlinear Transformation of a Lognormal Variable

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13. ABSTRACT ***(Maximum 200 words)***

This annual report summarizes the technical effort and computer code developed during the first year. Several formulations for Probabilistic Finite Element Analysis (PFEA) are described with emphasis on the selected formulation. The strategies being implemented in the first-version computer code to perform linear, elastic PFEA is described. The results of a series of select Space Shuttle Main Engine (SSME) component surveys are presented. These results identify the critical components and provide the information necessary for probabilistic structural analysis. The report is issued in three volumes: Volume I - summarizes theory and strategies; Volume II - summarizes surveys of critical space shuttle main engine components; and Volume III - summarizes literature surveys and technical reports available at that time.

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